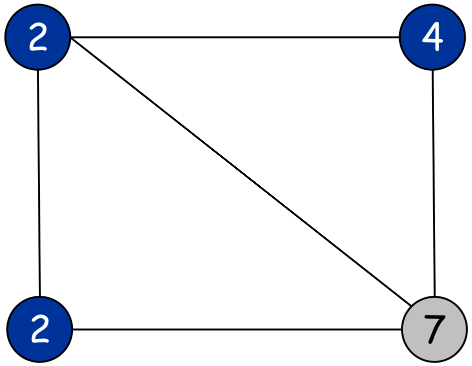
Kapitel 4:
Approximation Algorithms

Inhalt:

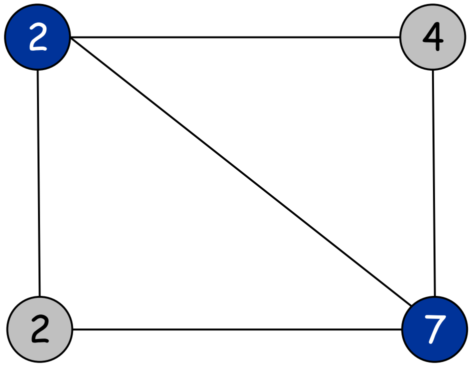
- Greedy Techniques
  - Load-Balancing Problem
  - Center Selection Problem
- Pricing Method
  - Vertex Cover Problem
- Linear Programming and Rounding
  - Vertex Cover Problem
  - Generalized Load-Balancing Problem
- Polynomial Time Approximation Scheme
  - Knapsack Problem

Weighted Vertex Cover

Weighted vertex cover. Given a graph G with vertex weights, find a vertex cover of minimum weight.



weight = 2 + 2 + 4 = 8



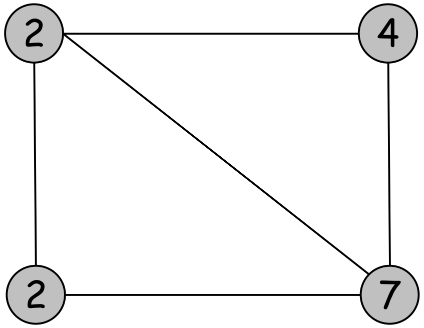
weight = 9

Weighted Vertex Cover

Pricing method. Each edge must be covered by some vertex i. Edge e pays price $p_e \geq 0$ to use vertex i.

Fairness. Edges incident to vertex i should pay $\leq w_i$ in total.

$$\text{for each vertex } i: \sum_{e=(i,j)} p_e \leq w_i$$



Claim. For any vertex cover S and any fair prices p_e: $\sum_e p_e \leq w(S)$.

Proof.

$$\sum_{e \in E} p_e \underset{\uparrow}{\leq} \sum_{i \in S} \sum_{e=(i,j)} p_e \underset{\uparrow}{\leq} \sum_{i \in S} w_i = w(S). \quad \blacksquare$$

each edge e covered by at least one node in S

sum fairness inequalities for each node in S

Pricing Method

Pricing method. Set prices and find vertex cover simultaneously.

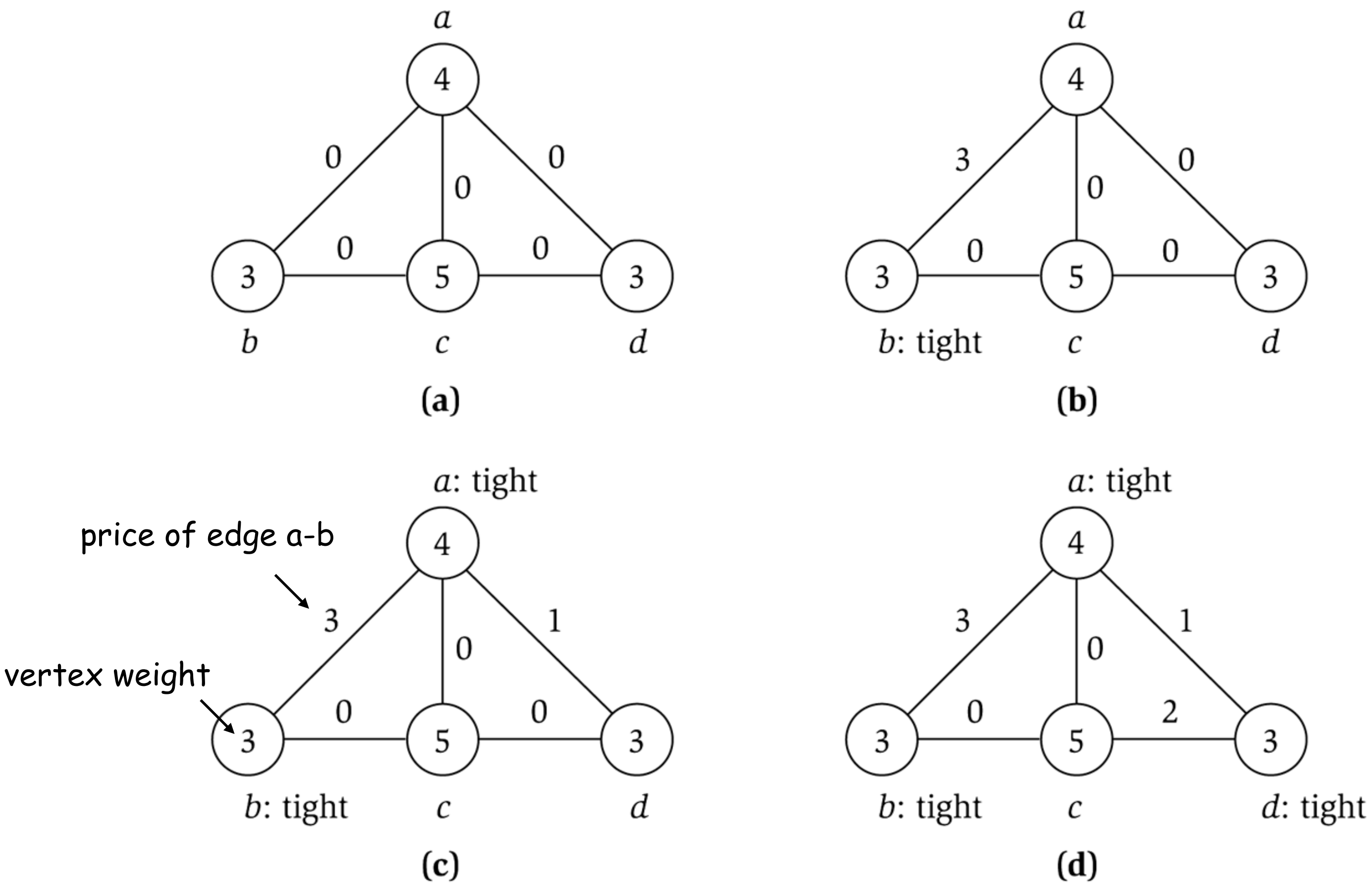
$$\sum_{e=(i,j)} p_e = w_i$$

```
Weighted-Vertex-Cover-Approx(G, w) {
   foreach e in E
      $p_e$ = 0

   while (∃ edge i-j such that neither i nor j are tight)
      select such an edge e
      increase $p_e$ without violating fairness
   }

   S ← set of all tight nodes
   return S
}
```

Pricing Method



Pricing Method: Analysis

Theorem. Pricing method is a 2-approximation.

Pf.

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.
- Let S = set of all tight nodes upon termination of algorithm. S is a vertex cover: if some edge i-j is uncovered, then neither i nor j is tight. But then while loop would not terminate.
- Let S* be optimal vertex cover. We show $w(S) \leq 2w(S^*)$.

$$w(S) = \sum_{i \in S} w_i \underset{\substack{\uparrow \\ \text{all nodes in S are tight}}}{=} \sum_{i \in S} \sum_{e=(i,j)} p_e \underset{\substack{\uparrow \\ S \subseteq V, \\ \text{prices} \geq 0}}{\leq} \sum_{i \in V} \sum_{e=(i,j)} p_e \underset{\substack{\uparrow \\ \text{each edge counted twice}}}{=} 2 \sum_{e \in E} p_e \underset{\substack{\uparrow \\ \text{fairness lemma}}}{\leq} 2w(S^*). \quad \blacksquare$$

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Integer Programming

INTEGER-PROGRAMMING.

Given integers a_{ij} and b_i, c_j
find integers x_j that satisfy:

$$
\begin{array}{llcl}
\max & c^t x & & \\
\text{s.t.} & Ax & \geq & b \\
 & x & & \text{integral}
\end{array}
$$

$$
\begin{array}{lcll}
\multicolumn{4}{l}{\max \sum_{j=1}^{n} c_j x_j} \\
\multicolumn{4}{l}{\text{s.t.}} \\
\sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\
x_j & \geq & 0 & 1 \leq j \leq n \\
x_j & & \text{integral} & 1 \leq j \leq n
\end{array}
$$

Linear Programming

Linear programming. Max/min linear objective function subject to linear inequalities.

- Input: integers c_j, b_i, a_{ij} .
- Output: real numbers x_j.

$$\begin{array}{llrcl} \text{(LP)} & \max & c^t x & & \\ & \text{s.t.} & Ax & \ge & b \\ & & x & \ge & 0 \end{array}$$

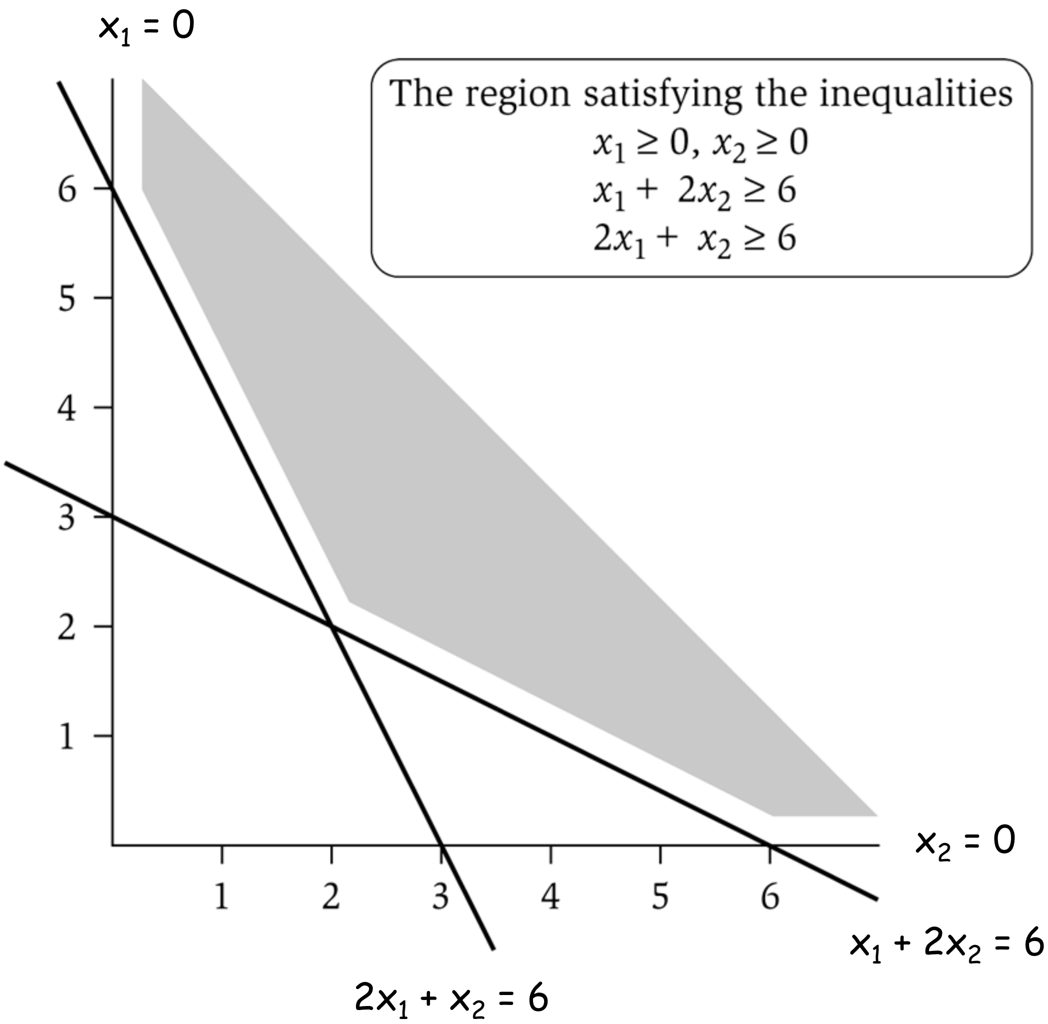
$$\begin{array}{llrcll} \text{(LP)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\ & \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \ge & b_i & 1 \le i \le m \\ & & x_j & \ge & 0 & 1 \le j \le n \end{array}$$

Linear. No x^2, xy, $\arccos(x)$, $x(1-x)$, etc.

Simplex algorithm. [Dantzig 1947] Can often solve LP in practice.
Ellipsoid algorithm. [Khachian 1979] Can solve LP in poly-time.
Interior Point Method. [Karmarkar 1984] Practical poly-time algorithm.

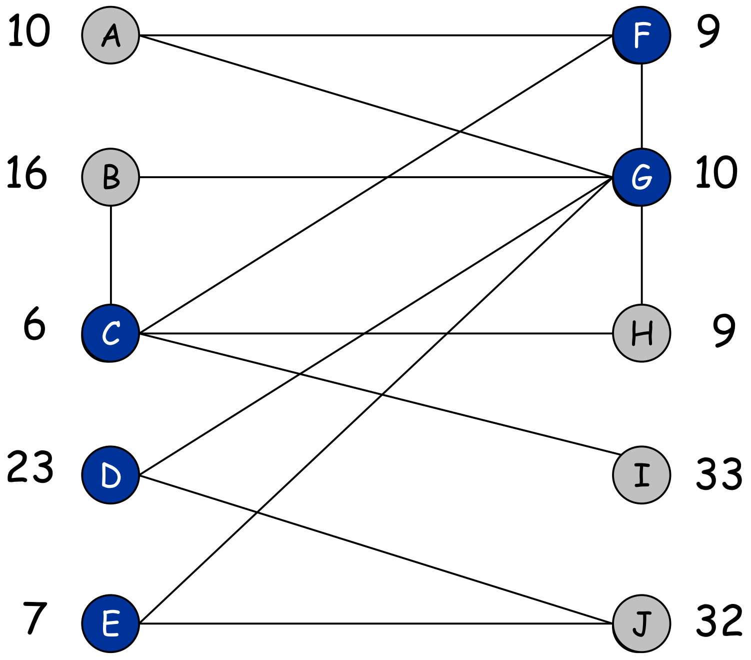
LP Feasible Region

LP geometry in 2D.



Weighted Vertex Cover

Weighted vertex cover. Given an undirected graph G = (V, E) with vertex weights $w_i \geq 0$, find a minimum weight subset of nodes S such that every edge is incident to at least one vertex in S.



total weight = 55

Weighted Vertex Cover: IP Formulation

Weighted vertex cover. Given an undirected graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a minimum weight subset of nodes S such that every edge is incident to at least one vertex in S.

Integer programming formulation.

- Model inclusion of each vertex i using a 0/1 variable x_i.

$$x_i = \begin{cases} 0 & \text{if vertex } i \text{ is not in vertex cover} \\ 1 & \text{if vertex } i \text{ is in vertex cover} \end{cases}$$

  Vertex covers in 1-1 correspondence with 0/1 assignments:
  $S = \{i \in V : x_i = 1\}$

- Objective function: minimize $\Sigma_i w_i x_i$.

- Must take either i or j for each edge (i,j): $x_i + x_j \geq 1$.

Weighted Vertex Cover: IP Formulation

Weighted vertex cover. Integer programming formulation.

$$
\begin{array}{llllll}
(ILP) & \min & \displaystyle\sum_{i \in V} w_i \, x_i & & & \\
& \text{s. t.} & x_i + x_j & \geq & 1 & (i, j) \in E \\
& & x_i & \in & \{0,1\} & i \in V
\end{array}
$$

Observation. If x* is optimal solution to (ILP), then $S = \{i \in V : x^*_i = 1\}$ is a min weight vertex cover.

Weighted Vertex Cover: LP Relaxation

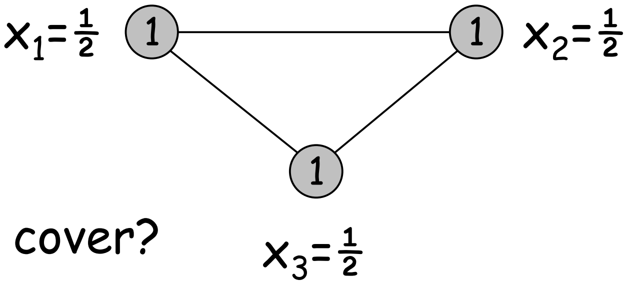
Weighted vertex cover. Linear programming formulation.

$$
\begin{array}{llll}
(LP)\ \min & \sum_{i \in V} w_i\, x_i & & \\
\text{s. t.} & x_i + x_j & \geq 1 & (i, j) \in E \\
 & x_i & \geq 0 & i \in V
\end{array}
$$

Observation. Optimal value of (LP) is $\leq$ optimal value of (ILP).

Pf. LP has fewer constraints.

Note. LP is not equivalent to vertex cover.



Q. How can solving LP help us find a small vertex cover?

A. Solve LP and round fractional values.

Weighted Vertex Cover

Theorem. If x* is optimal solution to (LP), then $S = \{i \in V : x^*_i \geq \frac{1}{2}\}$ is a vertex cover whose weight is at most twice the min possible weight.

Pf. [S is a vertex cover]

- Consider an edge $(i, j) \in E$.
- Since $x^*_i + x^*_j \geq 1$, either $x^*_i \geq \frac{1}{2}$ or $x^*_j \geq \frac{1}{2}$ $\Rightarrow$ (i, j) covered.

Pf. [S has desired cost]

- Let S* be optimal vertex cover. Then

$$\sum_{i \in S^*} w_i \underset{\substack{\uparrow \\ \text{LP is a relaxation}}}{\geq} \sum_{i \in S} w_i \, x_i^* \underset{\substack{\uparrow \\ x^*_i \geq \frac{1}{2}}}{\geq} \frac{1}{2} \sum_{i \in S} w_i$$

Weighted Vertex Cover

Theorem. [Hochbaum 1982] 2-approximation algorithm for weighted vertex cover.

Theorem. [Dinur-Safra 2001] If P $\neq$ NP, then no ρ-approximation for $\rho < 1.3607$, even with unit weights.

$\nwarrow$

$10 \sqrt{5} - 21$

Open research problem. Close the gap.

Integer Programming

INTEGER-PROGRAMMING.

Given integers a_{ij}, b_i, and c_i , find integers x_j that satisfy:

$$\begin{array}{llll} \max & c^t x & & \\ \text{s. t.} & Ax & \geq & b \\ & x & & \text{integral} \end{array} \quad \Leftrightarrow \quad \begin{array}{llll} \max & c^t x & & \\ \text{s.t.} & -Ax & \leq & -b \\ & x & & \text{integral} \end{array}$$

$$\begin{array}{llll} \max & c^t x & & \\ \text{s. t.} & Ax & \geq & b \\ & x & & \text{integral} \end{array} \quad \Leftrightarrow \quad \begin{array}{llll} \min & -c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & & \text{integral} \end{array}$$

$$\begin{array}{llll} \max & c^t x & & \\ \text{s.t.} & Ax & = & b \\ & x & & \text{integral} \end{array} \quad \Leftrightarrow \quad \begin{array}{llllll} \max & c^t x & & & & \\ \text{s.t.} & Ax & \leq & b & \wedge & Ax \geq b \\ & x & & & & \text{integral} \end{array}$$

Integer Programming

INTEGER-PROGRAMMING.

Given integers a_{ij}, b_i, and c_i , find integers x_j that satisfy:

$$\begin{array}{lccc} \min & c^t x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & & \text{integral} \end{array}$$

The primal dual problem is defined as follows:
Given integers a_{ij}, b_i, and c_i , find integers y_j that satisfy:

$$\begin{array}{lccc} \max & y^t b & & \\ \text{s.t.} & y^t A & \leq & c^t \\ & y & & \text{integral} \end{array}$$

Fragen?