# Premaster Course Algorithms 1

SS 2018

**Prof. Dr. Christian Scheideler**

# Basic Information

## Lectures:

- Mo 4 – 6 pm F1.110

## Exam:

- Oral exam at end of course

## Course Webpage:

- http://cs.uni-paderborn.de/ti/lehre/veranstaltungen/ss-2018/premaster-algorithms-1/

## Office hours:

- Thu, 4-5 pm, F2.326

# Basic Information

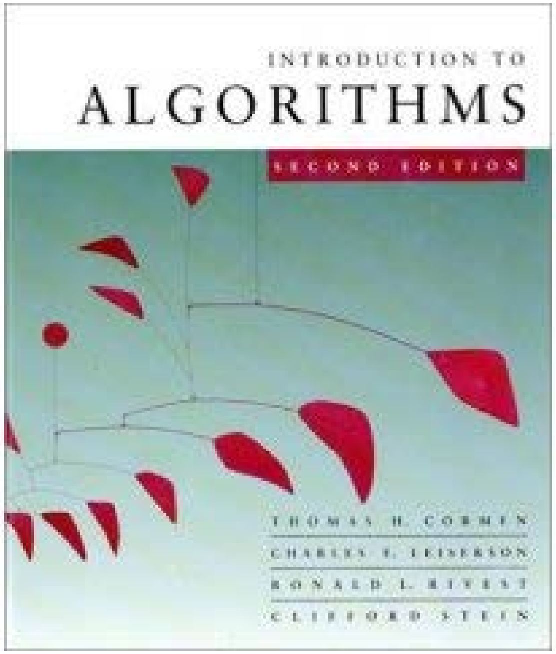
**Literature:**

Cormen, Leiserson, Rivest, Stein:

Introduction to Algorithms, 3rd ed.

MIT Press/McGraw-Hill

ISBN 0-262-53305-8



# Basic Information

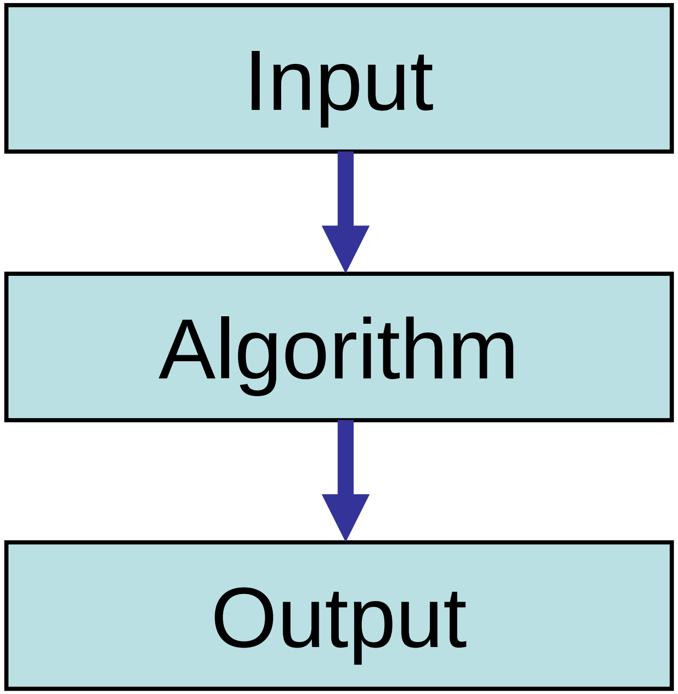
**Contents of the course:**

- April 9: Chapters 1-4 (Intro and runtime analysis)
- April 16: Chapters 6-7 (Sorting)
- April 23: Chapters 10-12
  (Elementary data structures)
- April 30: Chapters 15-16
  (Dynamic programming and greedy algorithms)
- May 7: Chapters 24-25 (Basic graph algorithms)
- May 14: Chapters 24-25 (Shortest paths)
- May 28: Chapter 26 (Maximum flow)

# What is an Algorithm?

*Definition 1.1*: An *algorithm* is a concise description of a procedure to solve a certain class of problems.

*Here:*



Input

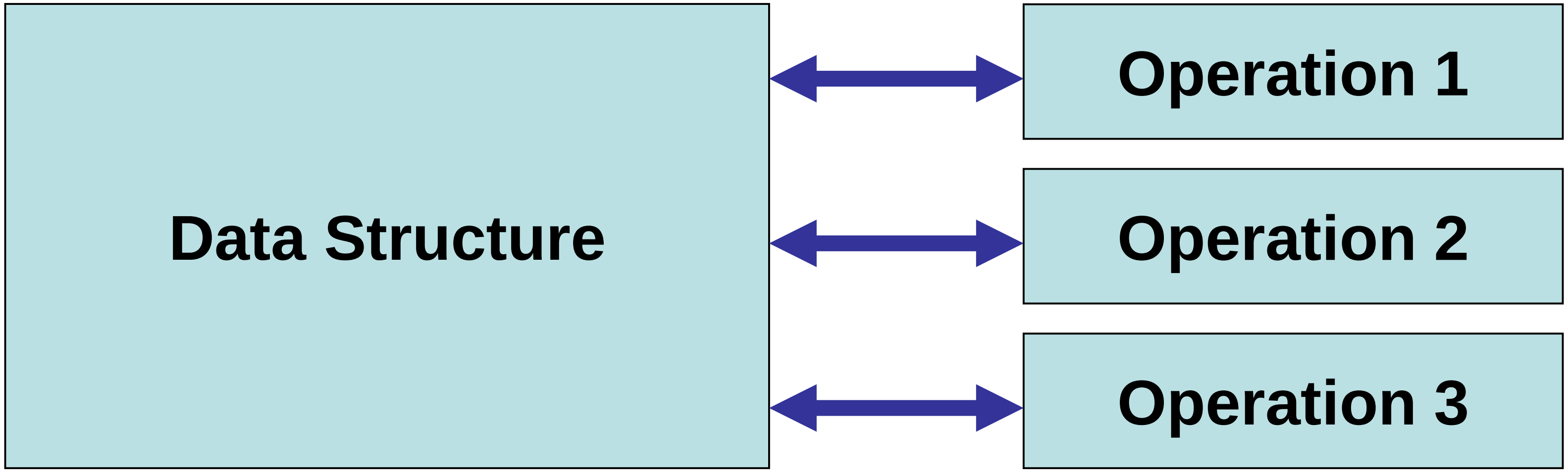
Algorithm

Output

# What is a Data Structure?

*Definition 1.2*: A *data structure* is a specific way of organizing data in the memory of a computer to facilitate operations like *Seach, Insert, and Delete*.

Here:



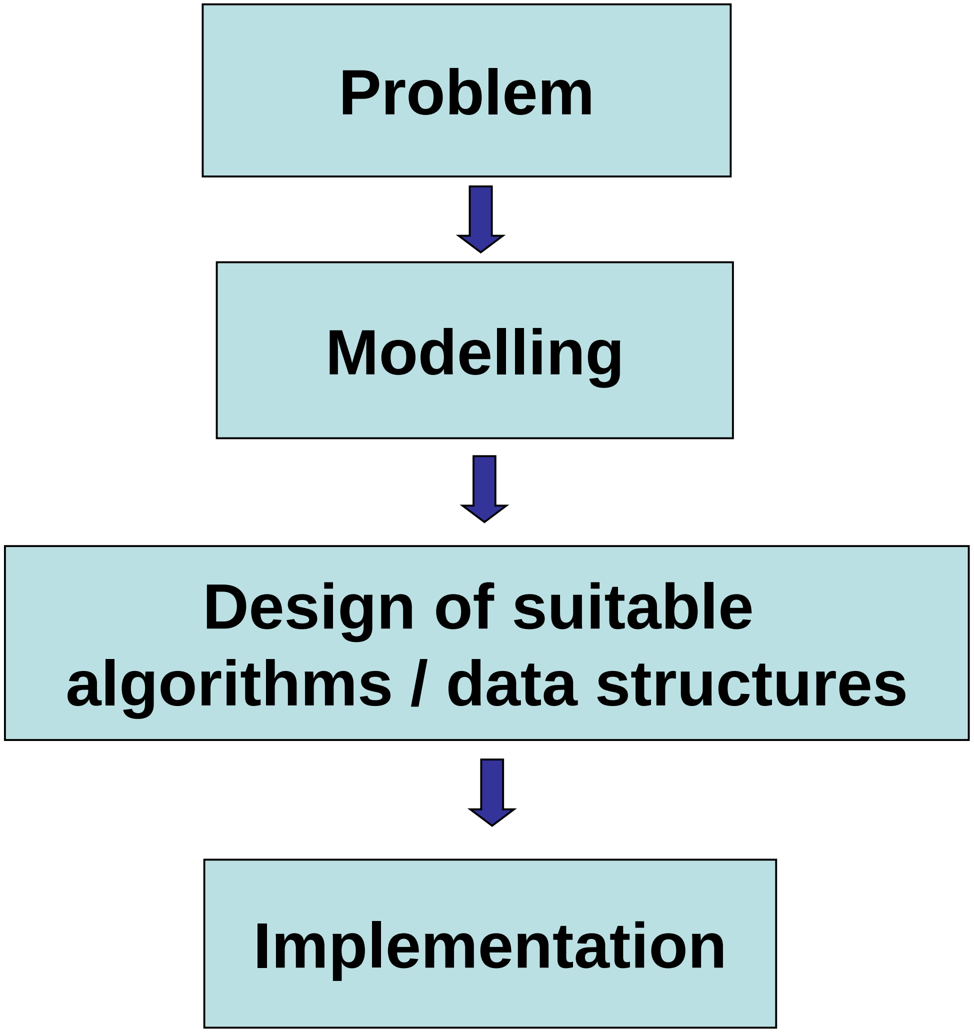
Data Structure

Operation 1

Operation 2

Operation 3

# Software Development



Problem

Modelling

Design of suitable algorithms / data structures

Implementation

# Important Criteria

- Algorithms / data structures must be correct.

  ⇒ *Correctness proofs*.

- Algorithms / data structures should work efficiently.

  ⇒ *Analytical methods for time and space analysis*.

- For *guarantees*, analytical methods *cannot* rely on empirical studies but must be based on a *mathematical analysis*.

# Design of Algorithms

A rigorous algorithmic approach requires:

1. *Formal description of algorithm (in pseudo-code)*
2. *Correctness proof*
3. *Formal time and/or space analysis*

# Notation

Pseudocode:

- Loops (for, while, repeat)
- Branching (if – then – else)
- Returning from procedure call (return)
- Assignment (:=)
- Block structure via indentation

Time needed for elementary operations (like assignments and checks): constant

# Example: Minimum Search

Input: sequence A of n numbers $(a_1,a_2,\ldots,a_n)$
Output: index i with $a_i \le a_j$ for all $1 \le j \le n$.

Algorithm:

Min-Search(A):

```
min:=1
for j:=2 to length(A)
    if A[j]<A[min] then min:=j
return min
```

Example:
Input: (31,41,59,26,51,48)
Output: 4

# Example: Sorting

Input: sequence A of n numbers $(a_1,a_2,\dots,a_n)$
Output: permutation of $(a_1,a_2,\dots,a_n)$ into $(b_1,b_2,\dots,b_n)$ with $b_1 \le b_2 \le \dots \le b_n$.

Algorithm:

Insertion-Sort(A):

```
for j:=2 to length(A)
    key:=A[j]; i:=j-1  // insert A[j] into A[1..j-1]
    while i>0 and A[i]>key
        A[i+1]:=A[i]
        i:=i-1
    A[i+1]:=key
return A
```

# Example: Sorting

Runtime analysis:

```
Insertion-Sort(A):                      cost      times
1  for j:=2 to length(A)                $c_1$     $n$
2      key:=A[j]; i:=j-1                $c_2$     $n-1$
3      while i>0 and A[i]>key           $c_3$     $\Sigma_{j=2}^{n} t_j$
4          A[i+1]:=A[i]                 $c_4$     $\Sigma_{j=2}^{n} (t_j-1)$
5          i:=i-1                       $c_5$     $\Sigma_{j=2}^{n} (t_j-1)$
6      A[i+1]:=key                      $c_6$     $n-1$
7  return A                             $c_7$     $1$
```

Worst case: $t_j=j$ (A[j] has to be placed into A[1])

Worst case runtime: $T(n) \approx c \cdot n^2$ for some constant c

# Example: Sorting

Input: sequence A of n numbers $(a_1,a_2,\ldots,a_n)$
Output: permutation of $(a_1,a_2,\ldots,a_n)$ into $(b_1,b_2,\ldots,b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$.

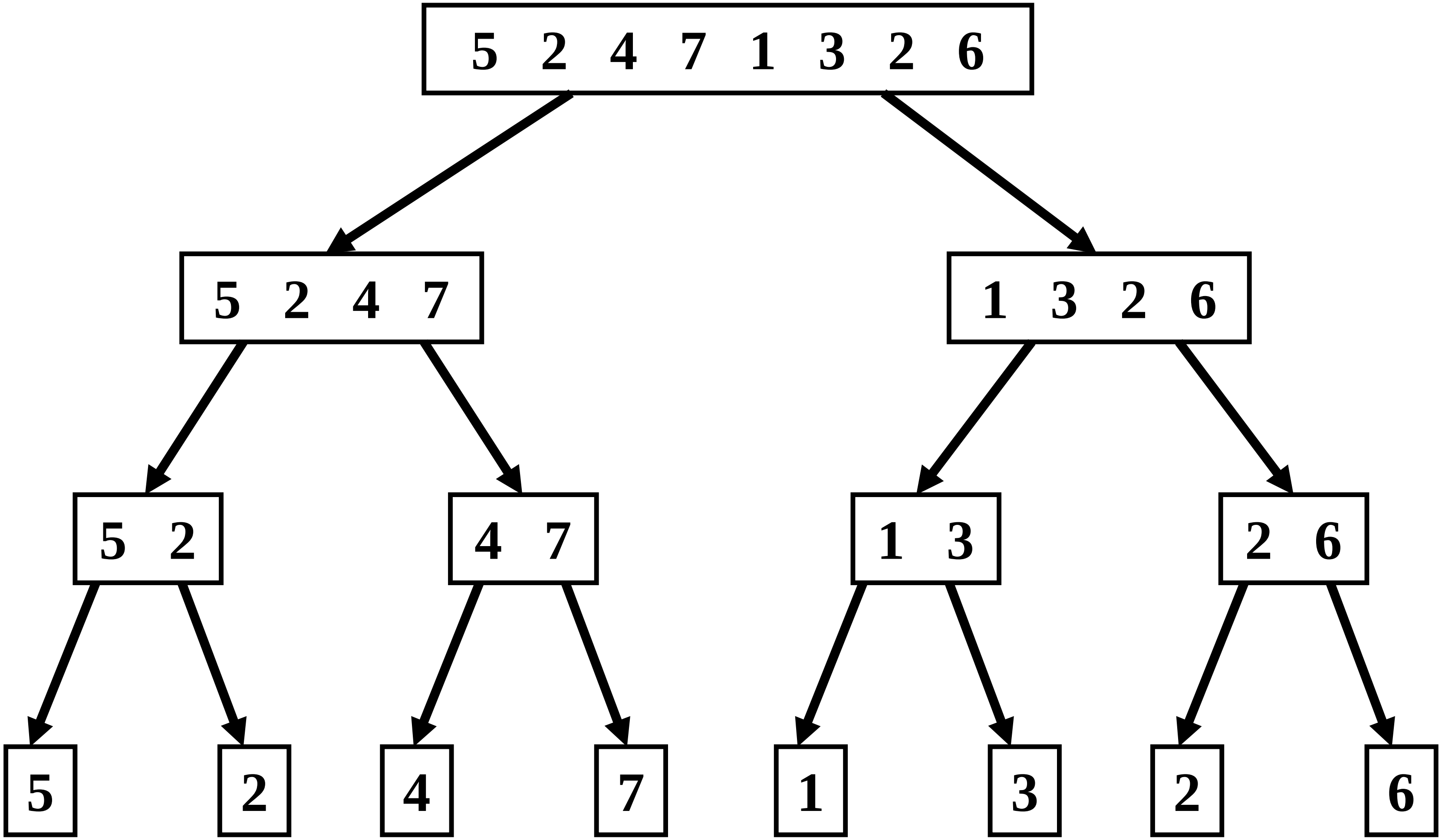
Algorithm: Merge-Sort(A,1,n)

Merge-Sort(A,p,r):

```
if p<r then
    q:=$\lfloor (p+r)/2 \rfloor$  // compute middle position
    Merge-Sort(A,p,q)
    Merge-Sort(A,q+1,r)
    Merge(A,p,q,r)
```

Merge(A,p,q,r) merges sorted A[p..q] and A[q+1…r] to sorted A[p..r].

# Illustration of Merge-Sort (1)



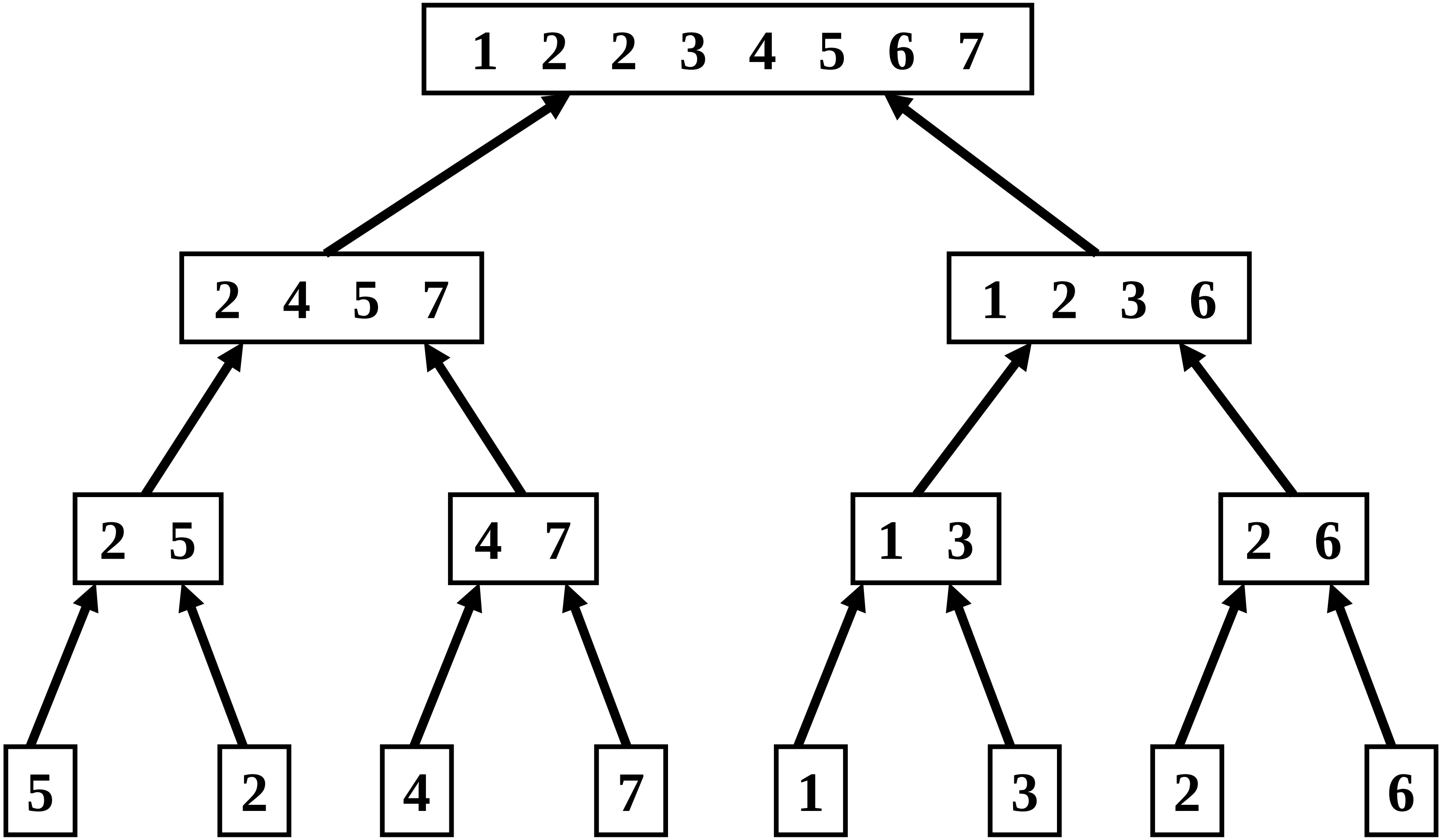
**5 2 4 7 1 3 2 6**

**5 2 4 7** **1 3 2 6**

**5 2** **4 7** **1 3** **2 6**

**5** **2** **4** **7** **1** **3** **2** **6**

# Illustration of Merge-Sort (2)



# Example: Sorting

Runtime analysis: ($n=r-p+1$: number of elements)

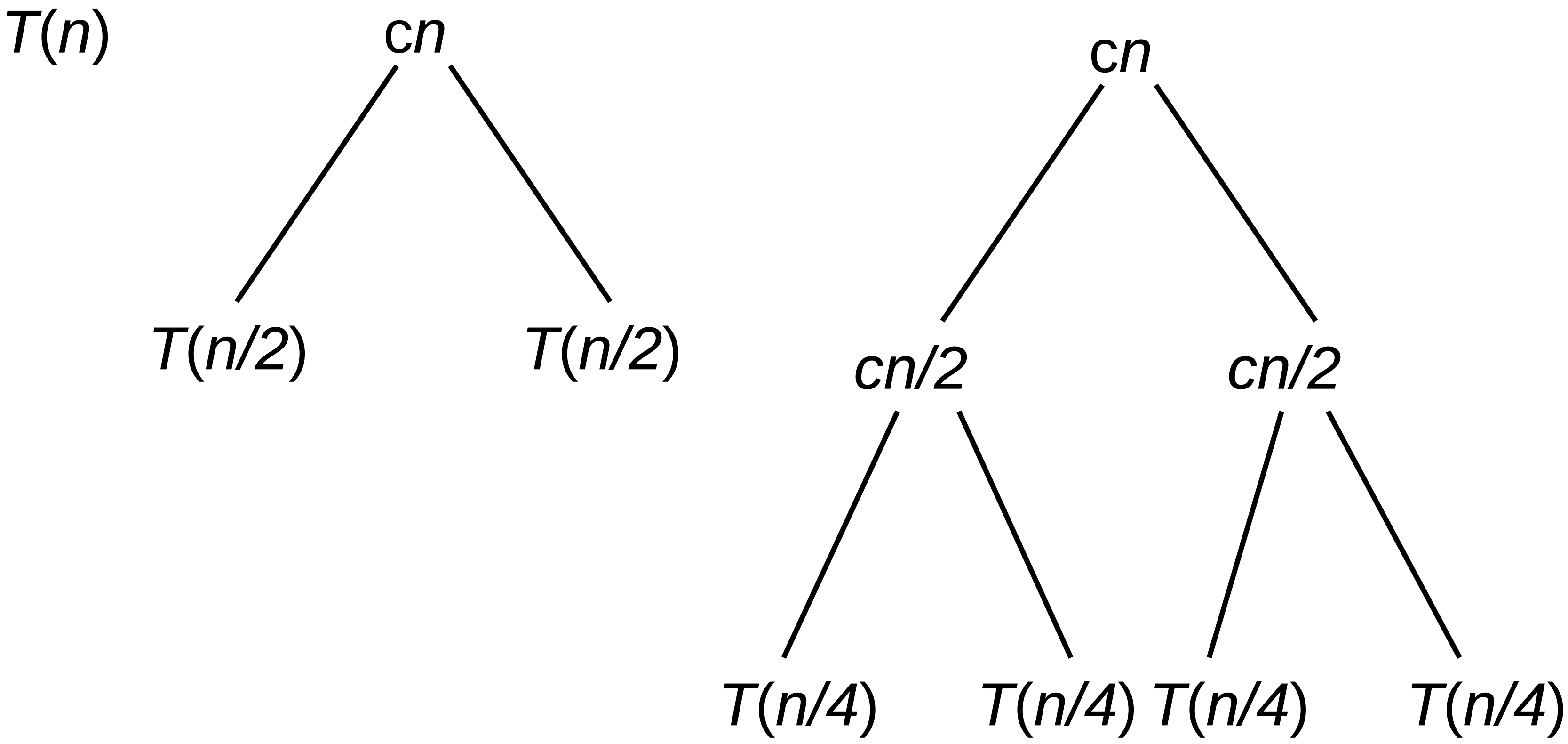
| Merge-Sort(A,p,r): | cost | times |
|---|---|---|
| **if** p<r **then** | $c_1$ | 1 |
|     $q:=\lfloor (p+r)/2 \rfloor$ | $c_2$ | 1 |
|     Merge-Sort(A,p,q) | $T(q-p+1)$ | 1 |
|     Merge-Sort(A,q+1,r) | $T(r-q+1)$ | 1 |
|     Merge(A,p,q,r) | $c_3 \cdot n$ | 1 |

Merge(A,p,q,r) merges sorted A[p..q] and A[q+1…r] to sorted A[p..r]. Suppose that n is a power of 2. Then:

Runtime:

$$T(n) = \begin{cases} 2\cdot T(n/2)+c_3\cdot n+c_1+c_2 & \text{if } n>1 \\ c_1 & \text{if } n=1 \end{cases}$$

# Runtime of Merge-Sort



$T(n)$

$cn$

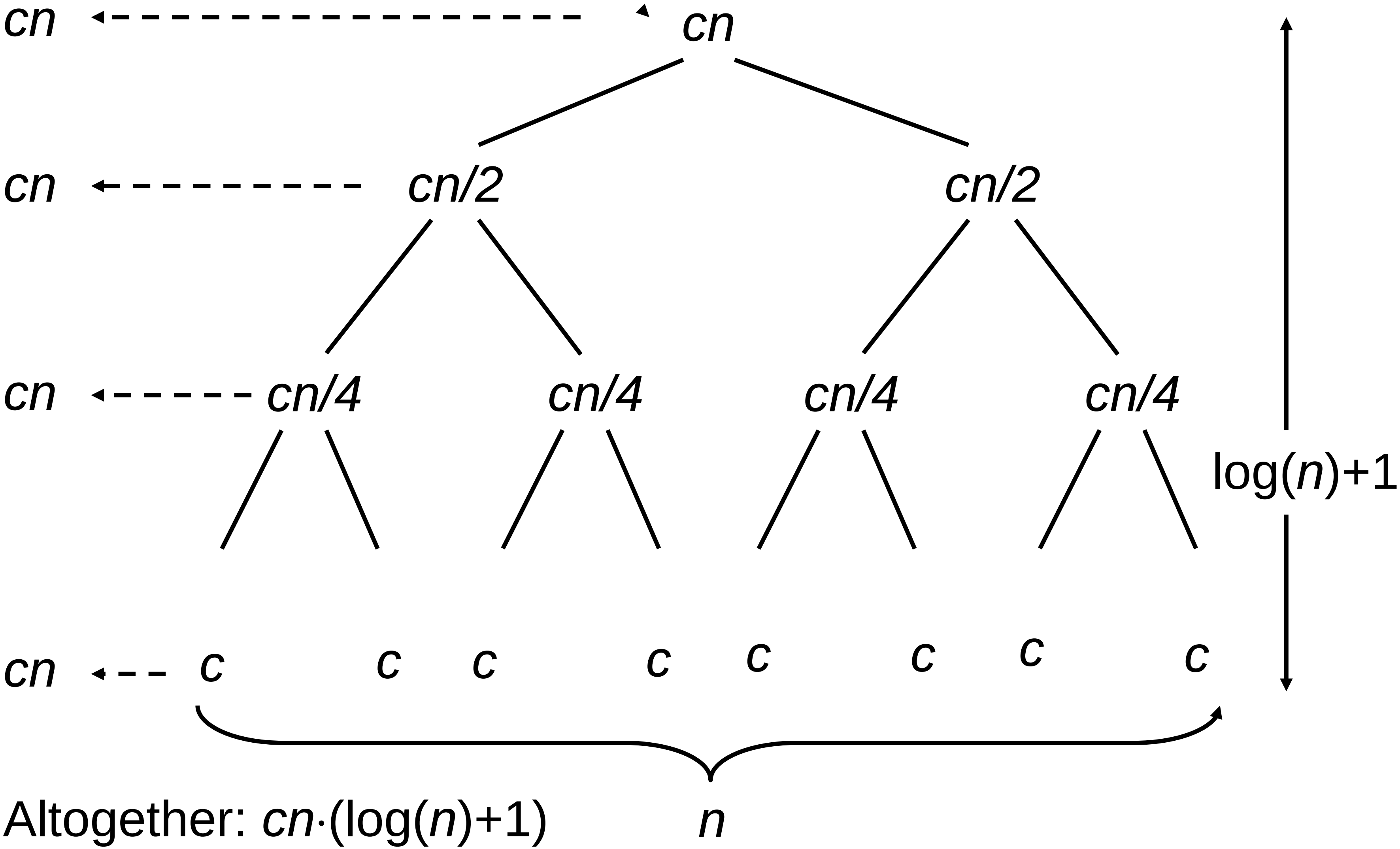
$T(n/2)$ $T(n/2)$

$cn$

$cn/2$ $cn/2$

$T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$

# Runtime of Merge-Sort



$cn$ ← $cn$

$cn$ ← $cn/2$ $cn/2$

$cn$ ← $cn/4$ $cn/4$ $cn/4$ $cn/4$

$cn$ ← $c$ $c$ $c$ $c$ $c$ $c$ $c$ $c$

$\log(n)+1$

$n$

Altogether: $cn\cdot(\log(n)+1)$

# Asymptotic Notation

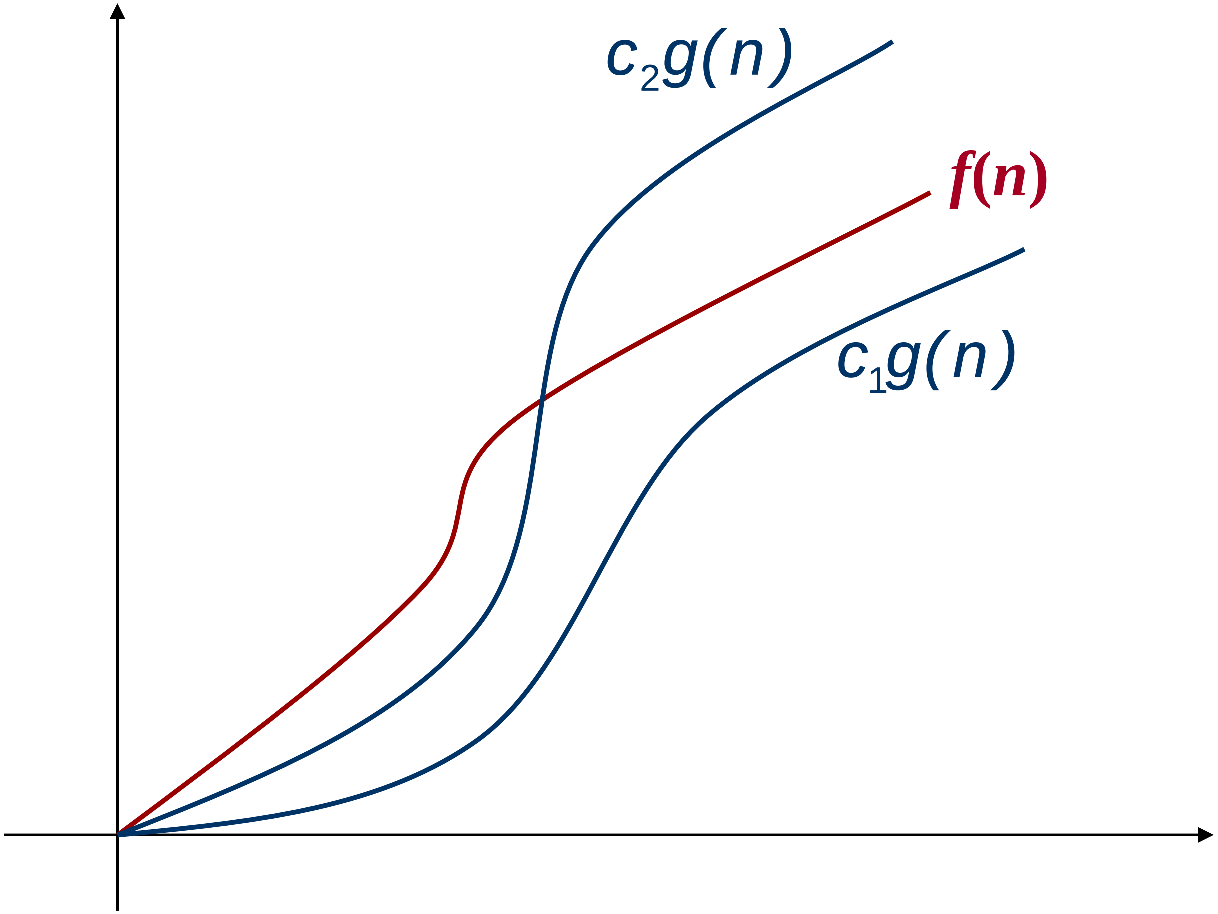
In theory, often just the asymptotic runtime of algorithms is of interest.

Asymptotic runtime ignores constants and just specifies the growth of functions.

Instead of $T(n)=c_1 \cdot n^2+c_2 \cdot n-c_3$ just $T(n)=\Theta(n^2)$.
( $\Theta$: „grows as fast as“ )

Formally,
$\Theta(g(n))$ = { f(n) | there exist positive constants $c_1 \le c_2$ and $n_0$ such that for all $n \ge n_0$, $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ }

# Illustration of Θ(*g*(*n*))



# Asymptotic Notation

$\Theta$(g(n)) = { f(n) | there exist positive constants $c_1 \leq c_2$ and $n_0$ such that for all $n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ }

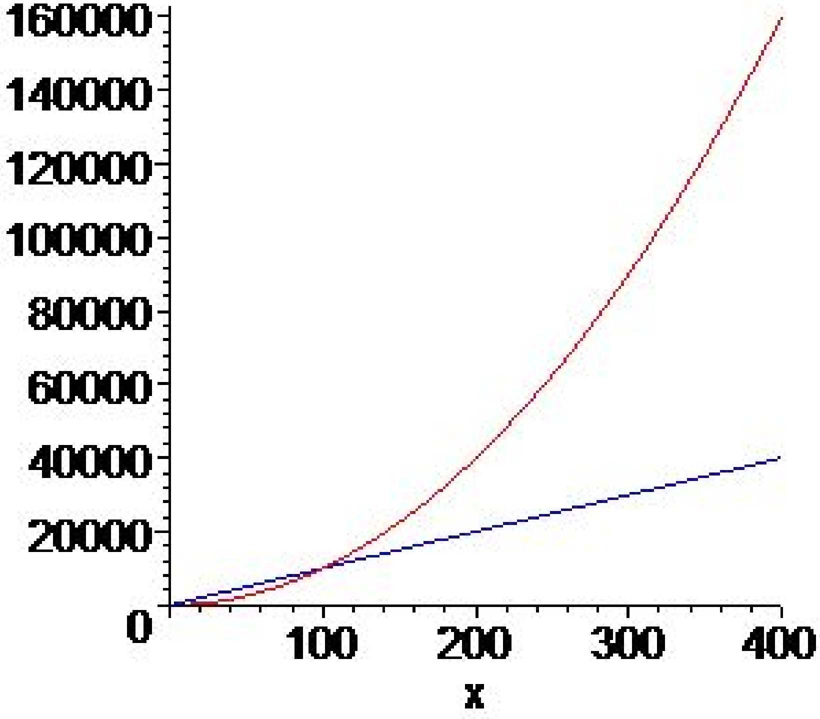
O(g(n)) = { f(n) | there exist positive constants c and $n_0$ such that for all $n \geq n_0$, $0 \leq f(n) \leq c \cdot g(n)$ }

$\Omega$(g(n)) = { f(n) | there exist positive constants c and $n_0$ such that for all $n \geq n_0$, $0 \leq c \cdot g(n) \leq f(n)$ }

If $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n)$ then $f(n) \in \Theta(g(n))$.

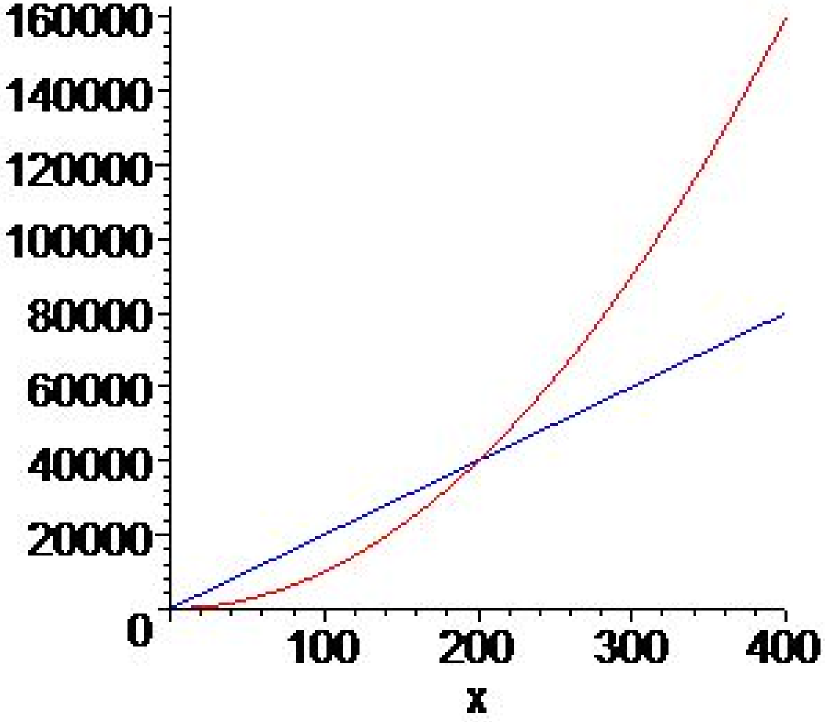
By abuse of notation, one often writes $f(n)=O(g(n))$ instead of $f(n) \in O(g(n))$ .

# Illustration of O(*g*(*n*))



$g(x)=x^2$

$f(x)=100x$



$g(x)=x^2$

$f(x)=200x$

# Asymptotic Notation

Transitity:

- If $f(n)=\Theta(g(n))$ and $g(n)=\Theta(h(n))$ then $f(n)=\Theta(h(n))$.
- If $f(n)=O(g(n))$ and $g(n)=O(h(n))$ then $f(n)=O(h(n))$.
- If $f(n)=\Omega(g(n))$ and $g(n)=\Omega(h(n))$ then $f(n)=\Omega(h(n))$.

Reflexivity:

- $f(n)=\Theta(f(n))$, $f(n)=O(f(n))$, and $f(n)=\Omega(f(n))$

Symmetry:

- $f(n)=\Theta(g(n))$ if and only if $g(n)=\Theta(f(n))$.

Transpose symmetry:

- $f(n)=O(g(n))$ if and only if $g(n)=\Omega(f(n))$

# Asymptotic Notation

Theorem 1.3: Let $p(n) = \sum_{i=0}^{k} a_i \cdot n^i$ with $a_k>0$. Then we have $p(n) = \Theta(n^k)$.

Proof:

To show: $p(n) = O(n^k)$ and $p(n) = \Omega(n^k)$.

- $p(n) = O(n^k)$ : For all $n \geq 1$,

  $$p(n) \leq \sum_{i=0}^{k} |a_i|\, n^i \leq n^k \sum_{i=0}^{k} |a_i|$$

- Hence, definition of $O()$ is satisfied with $c=\sum_{i=0}^{k} |a_i|$ and $n_0=1$.

- $p(n) = \Omega(n^k)$ : For all $n \geq 2k \cdot A/a_k$ and $A=\max_i |a_i|$,

  $p(n) \geq a_k \cdot n^k - \sum_{i=0}^{k-1} A \cdot n^i \geq a_k\, n^k - k \cdot A\, n^{k-1} \geq a_k\, n^k/2$
- Hence, definition of $\Omega()$ is satisfied with $c=a_k/2$ and $n_0=2kA/a_k$.

# Asymptotic Runtime Analysis

Worst-case runtime:

- T(I): worst-case runtime of instruction I
- T(elementary command) = O(1)
- T(return x) = O(1)
- T(I;I´) = T(I) + T(I´)
- T(if C then I else I´) = T(C) + max{T(I),T(I´)}
- T(for i:=a to b do I) = $\sum_{i=a}^{b}$ (O(1)+T(I))
- T(repeat I until C) = $\sum_{i=1}^{k}$ (T(C)+T(I))
  (k: number of iterations)
- T(while C do I) = $\sum_{i=1}^{k}$ (T(C)+T(I))

Runtime analysis difficult for while- und repeat-loops since we need to determine k, which is sometimes not so easy!

# Asymptotic Runtime Analysis

Worst-case runtime:

- T(I): worst-case runtime of instruction I
- T(elementary command) = O(1)
- T(return x) = O(1)
- T(I;I´) = T(I) + T(I´)
- T(if C then I else I´) = T(C) + max{T(I),T(I´)}
- T(for i:=a to b do I) = $\sum_{i=a}^{b}$ (O(1)+T(I))
- T(repeat I until C) = $\sum_{i=1}^{k}$ (T(C)+T(I))
  (k: number of iterations)
- T(while C do I) = $\sum_{i=1}^{k}$ (T(C)+T(I))

To evaluate O-expressions we make use of the rule that for any positive functions f and g, O(f(n)) + O(g(n)) = O(f(n)+g(n)).

# Example: Computation of Sign

Input: number $x \in \mathbb{R}$

Signum(x):

```
if $x<0$ then return $-1$          $O(1)$
if $x>0$ then return $1$           $O(1)$
return $0$                         $O(1)$
```

runtime: $O(1)+O(1)+O(1) = O(1+1+1)$

$= O(1)$

# Example: Minimum

Input: array of numbers $A[1],\ldots,A[n]$

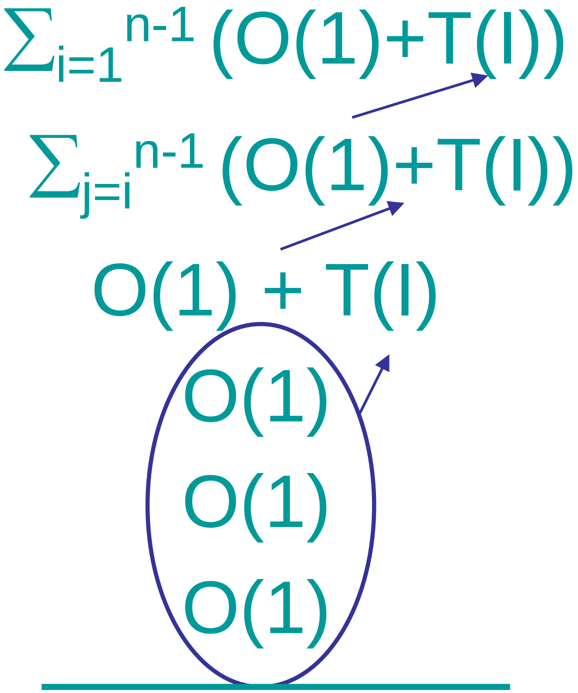
```
Minimum(A):
min := 1                                  $O(1)$
for i:=1 to n do                          $\sum_{i=1}^{n} (O(1)+T(I))$
    if A[i]<min then min:=A[i]            $O(1)$
return min                                $O(1)$
```

runtime: $O(1) + \sum_{i=1}^{n} O(1) + O(1) = O(2+\sum_{i=1}^{n} 1)$
$= O(n)$

# Example: Sorting

Input: array of numbers A[1],….,A[n]

```
Bubblesort(A):
for i:=1 to n-1 do
    for j:= n-1 downto i do
        if A[j]>A[j+1] then
            x:=A[j]
            A[j]:=A[j+1]
            A[j+1]:=x
```



$\sum_{i=1}^{n-1} (O(1)+T(I))$

$\sum_{j=i}^{n-1} (O(1)+T(I))$

$O(1) + T(I)$

$O(1)$

$O(1)$

$O(1)$

runtime: $O(\sum_{i=1}^{n-1} \sum_{j=i}^{n-1} 1) = O(n^2)$

# Master Theorem

Theorem 1.4: For some positive constants $a,b,c$ and $d$ and $n=b^k$ for some natural number $k$ let

$$T(n) = a \cdot T(n/b) + c \cdot n \qquad \text{if } n>1$$

$$T(n) = d \qquad \text{if } n \leq 1$$

Then it holds that

$$T(n) = \Theta(n) \qquad \text{if } a<b$$

$$T(n) = \Theta(n \log n) \qquad \text{if } a=b$$

$$T(n) = \Theta(n^{\log_b a}) \qquad \text{if } a>b$$

# Master Theorem

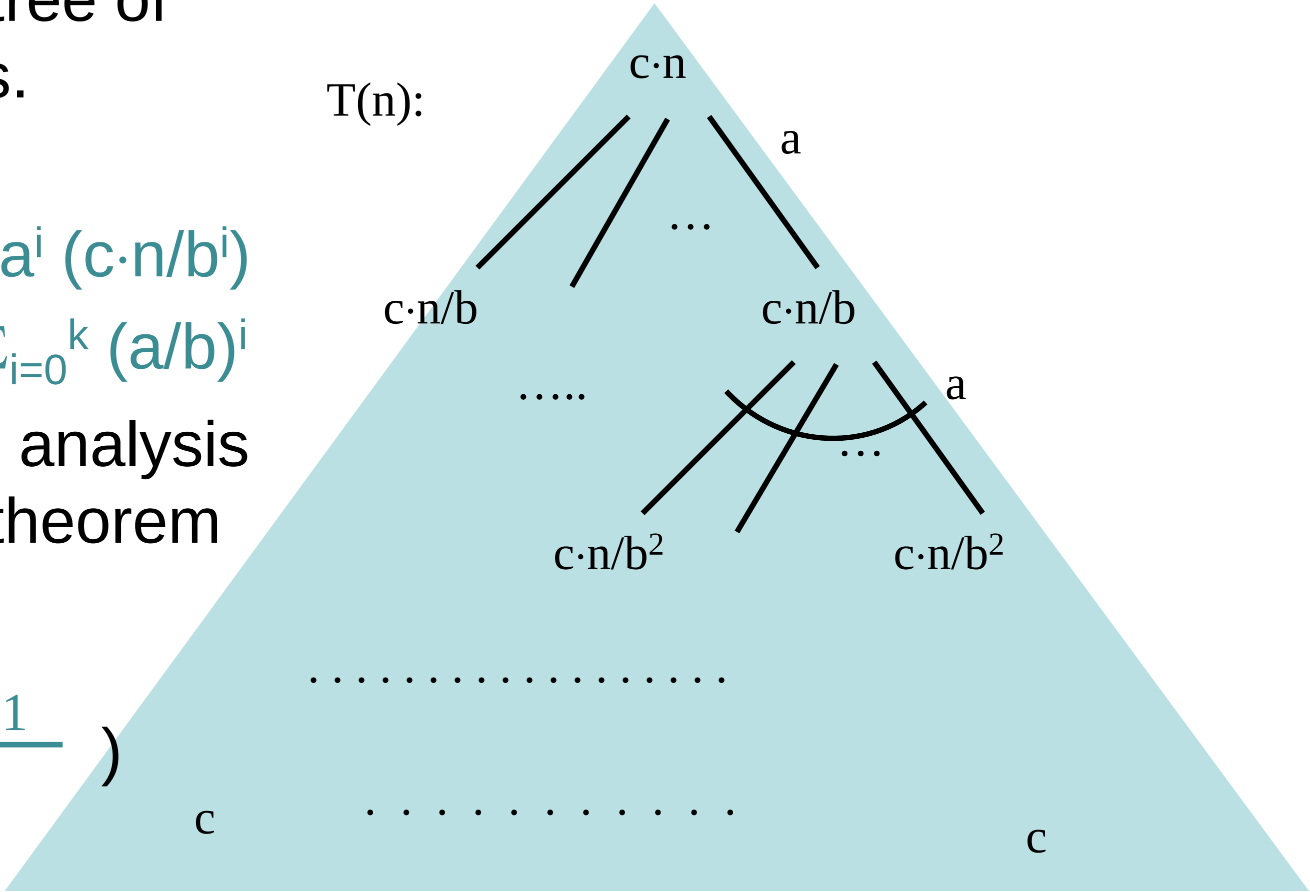
Proof:

- Consider the tree of recursive calls.
- It holds:

$$T(n) \leq \Sigma_{i=0}^{k} a^i (c\cdot n/b^i)$$
$$\leq c\cdot n \, \Sigma_{i=0}^{k} (a/b)^i$$

- Case-by-case analysis results in the theorem (use for $a\neq b$

$$\Sigma_{i=0}^{k} z^i = \frac{z^{i+1}-1}{z-1}$$ )



# General Master Theorem

Theorem 1.5: For some positive constants $a,b,d$, function $f(n)$ and $n=b^k$ for some integer $k$ let

$$T(n) = a \cdot T(n/b) + f(n) \qquad \text{if } n>1$$

$$T(n) = d \qquad \text{if } n \leq 1$$

Then it holds

- If $f(n)=O(n^{\log_b a-\varepsilon})$ for some $\varepsilon>0$ then $T(n) = O(n^{\log_b a})$.
- If $f(n)=O(n^{\log_b a})$ then $T(n) = O(n^{\log_b a} \cdot \log n)$.
- If $f(n)=O(n^{\log_b a+\varepsilon})$ for some $\varepsilon>0$ and $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c<1$ then $T(n)=O(f(n))$.

# Example: Matrix Multiplication

$$\begin{pmatrix} 3 & 7 & 5 & 4 \\ 0 & 3 & 2 & 4 \\ 10 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 \\ 2 & 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{pmatrix}$$

# Matrix Multiplication

$A=(a_{ij})_{1\leq i,j\leq n}$ $B=(b_{ij})_{1\leq i,j\leq n}$ $C=(c_{ij})_{1\leq i,j\leq n}$

$$\begin{pmatrix} 3 & 7 & 5 & 4 \\ 0 & 3 & 2 & 4 \\ 10 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 \\ 2 & 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 29 & 20 & 23 & 29 \\ 14 & 14 & \ldots & \\ \ldots & & & \\ \ldots & & & \end{pmatrix}$$

$c_{ij}= \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

# Matrix Multiplication

Problem: Compute product of two n×n matrices

- Input: matrices X,Y
- Output: matrix Z = X·Y

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \\ y_{4,1} & y_{4,2} & y_{4,3} & y_{4,4} \end{pmatrix}$$

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)
1.  **new** array Z[1,..,n][1,..,n]
2.  **for** i ← 1 **to** n **do**
3.      **for** j ← 1 **to** n **do**
4.          Z[i][j] ← 0
5.          **for** k ← 1 **to** n **do**
6.              Z[i][j] ← Z[i][j] + X[i][k] · Y[k][j]
7.  **return** Z
```

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)
    new array Z[1,..,n][1,..,n]
    for i ← 1 to n do
        for j ← 1 to n do
            $Z[i][j] \leftarrow 0$
            for k ← 1 to n do
                $Z[i][j] \leftarrow Z[i][j] + X[i][k] \cdot Y[k][j]$
    return Z
```

Runtime:

$\Theta(n^2)$

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)                          Runtime:
    new array Z[1,..,n][1,..,n]                              $\Theta(n^2)$
    for i ← 1 to n do                                        $\Theta(n)$
        for j ← 1 to n do
            Z[i][j] ← 0
            for k ← 1 to n do
                Z[i][j] ← Z[i][j] + X[i][k] · Y[k][j]
    return Z
```

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)                       Runtime:
new array Z[1,..,n][1,..,n]                               $\Theta(n^2)$
for i ← 1 to n do                                         $\Theta(n)$
    for j ← 1 to n do                                     $\Theta(n^2)$
        Z[i][j] ← 0
        for k ← 1 to n do
            Z[i][j] ← Z[i][j] + X[i][k] · Y[k][j]
return Z
```

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)                              Runtime:
    new array Z[1,..,n][1,..,n]                                  $\Theta(n^2)$
    for $i \leftarrow 1$ to n do                                 $\Theta(n)$
        for $j \leftarrow 1$ to n do                             $\Theta(n^2)$
            $Z[i][j] \leftarrow 0$                               $\Theta(n^2)$
            for $k \leftarrow 1$ to n do
                $Z[i][j] \leftarrow Z[i][j] + X[i][k] \cdot Y[k][j]$
    return Z
```

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)                              Runtime:
new array Z[1,..,n][1,..,n]                                      $\Theta(n^2)$
for i←1 to n do                                                  $\Theta(n)$
    for j ← 1 to n do                                            $\Theta(n^2)$
        Z[i][j] ← 0                                              $\Theta(n^2)$
        for k ←1 to n do                                         $\Theta(n^3)$
            Z[i][j] ← Z[i][j] + X[i][k] · Y[k][j]
return Z
```

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)                              Runtime:
new array Z[1,..,n][1,..,n]                                      $\Theta(n^2)$
for i ← 1 to n do                                                $\Theta(n)$
    for j ← 1 to n do                                            $\Theta(n^2)$
        Z[i][j] ← 0                                              $\Theta(n^2)$
        for k ← 1 to n do                                        $\Theta(n^3)$
            Z[i][j] ← Z[i][j] + X[i][k] · Y[k][j]                $\Theta(n^3)$
return Z
```

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)                            Runtime:
    new array Z[1,..,n][1,..,n]                                $\Theta(n^2)$
    for i←1 to n do                                            $\Theta(n)$
        for j ← 1 to n do                                      $\Theta(n^2)$
            Z[i][j] ← 0                                        $\Theta(n^2)$
            for k ←1 to n do                                   $\Theta(n^3)$
                Z[i][j] ← Z[i][j] + X[i][k] · Y[k][j]          $\Theta(n^3)$
    return Z                                                   $\Theta(1)$
```

# Matrix Multiplication

```
MatrixMultiplication(Array X, Y, n)                              Runtime:
    new array Z[1,..,n][1,..,n]                                  $\Theta(n^2)$
    for i←1 to n do                                              $\Theta(n)$
        for j ← 1 to n do                                        $\Theta(n^2)$
            Z[i][j] ← 0                                          $\Theta(n^2)$
            for k ←1 to n do                                     $\Theta(n^3)$
                Z[i][j] ← Z[i][j] + X[i][k] · Y[k][j]            $\Theta(n^3)$
    return Z                                                     $\Theta(1)$
                                                                 ____________
                                                                 $\Theta(n^3)$
```

# Matrix Multiplication

## Recursive approach:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}$$

## Recursive calls:

- 8 multiplications of $n/2 \times n/2$ matrices
- 4 additions of $n/2 \times n/2$ matrices

# Matrix Multiplication

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## Recursive calls:

- 8 multiplications of $n/2 \times n/2$ matrices
- 4 additions of $n/2 \times n/2$ matrices

## Runtime:

- $T(n) = 8 \cdot T(n/2) + \Theta(n^2)$

# Matrix Multiplication

## Runtime:

- $T(n) = \underset{\substack{\uparrow \\ \mathbf{a}}}{8} \cdot T(\underset{\substack{\uparrow \\ \mathbf{b}}}{n/2}) + \underset{\substack{\uparrow \\ \mathbf{f(n)}}}{k \cdot n^2}$

## General Master Theorem:

- $f(n) = k \cdot n^2$
- $a=8$, $b=2$
- Case 1: Runtime $\Theta(n^{\log_b a}) = \Theta(n^3)$
- Not better than elementary algorithm!

# Matrix Multiplication

## Strassen‘s Algorithm:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix}$$

## Trick:

$P_1 = A\cdot(F-H)$

$P_2 = (A+B)\cdot H$

$P_3 = (C+D)\cdot E$

$P_4 = D\cdot(G-E)$

$P_5 = (A+D)\cdot(E+H)$

$P_6 = (B-D)\cdot(G+H)$

$P_7 = (A-C)\cdot(E+F)$

$AE+BG = P_5 + P_4 - P_2 + P_6$

$AF+BH = P_1 + P_2$

$CE+DG = P_3 + P_4$

$CF+DH = P_5 + P_1 - P_3 - P_7$

**7 Multiplications!!!**

# Matrix Multiplication

## Runtime:

- $T(n) = \underset{\overset{\uparrow}{a}}{7} \cdot T(n/\underset{\overset{\uparrow}{b}}{2}) + \underset{\overset{\uparrow}{f(n)}}{k \cdot n^2}$

## General Master Theorem:

- $f(n) = k \cdot n^2$
- $a=7$, $b=2$
- Case 1: Runtime $\Theta(n^{\log_b a})=\Theta(n^{\log_2 7}) = \Theta(n^{2.81})$
- Better than elementary approach!

# Summary

We covered:

- Pseudo code
- Asymptotic runtime analysis

Next lecture:

- Sorting