Fundamental Algorithms

Chapter 3: Advanced Search Structures

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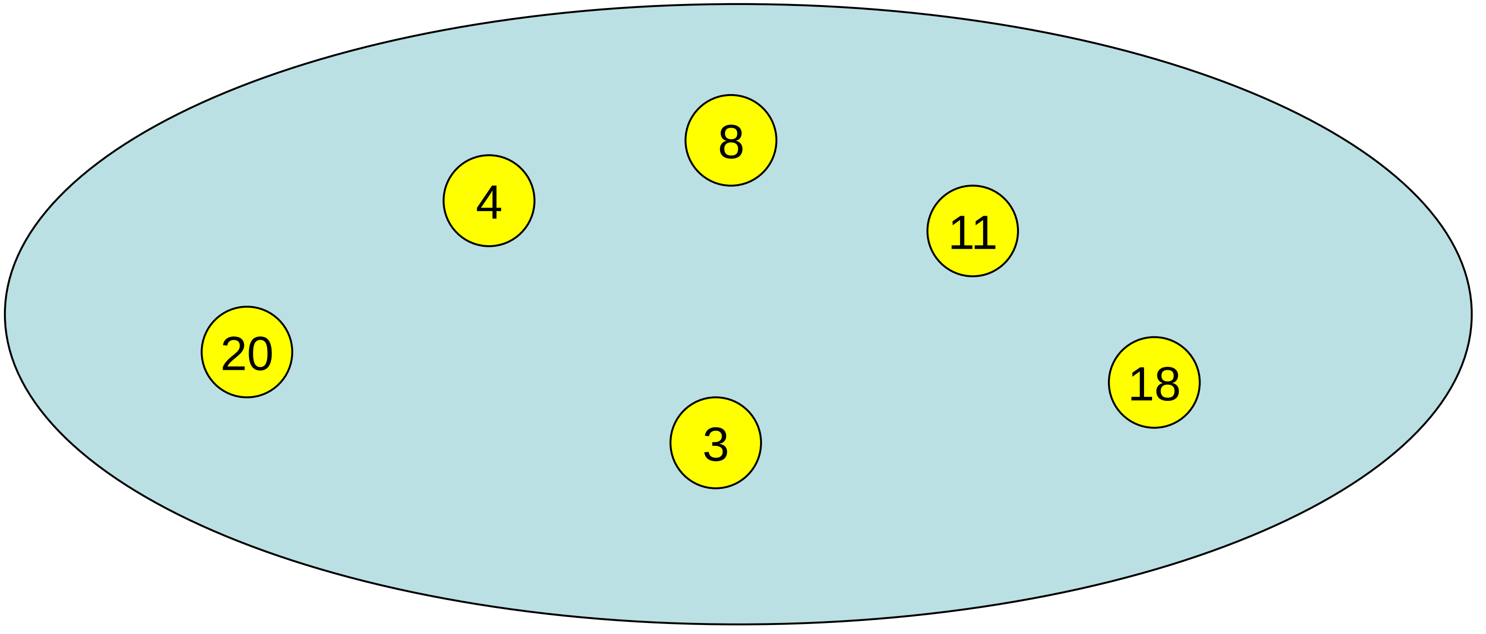
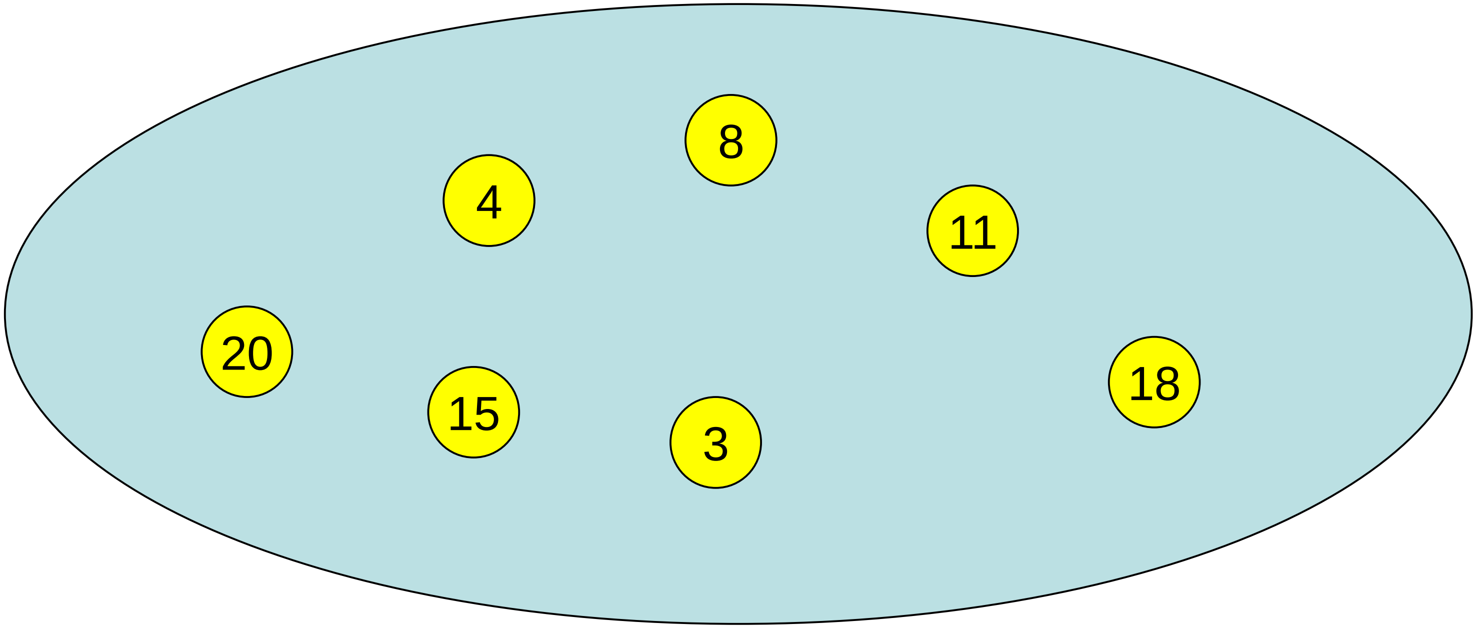
WS 2017

Search Structure



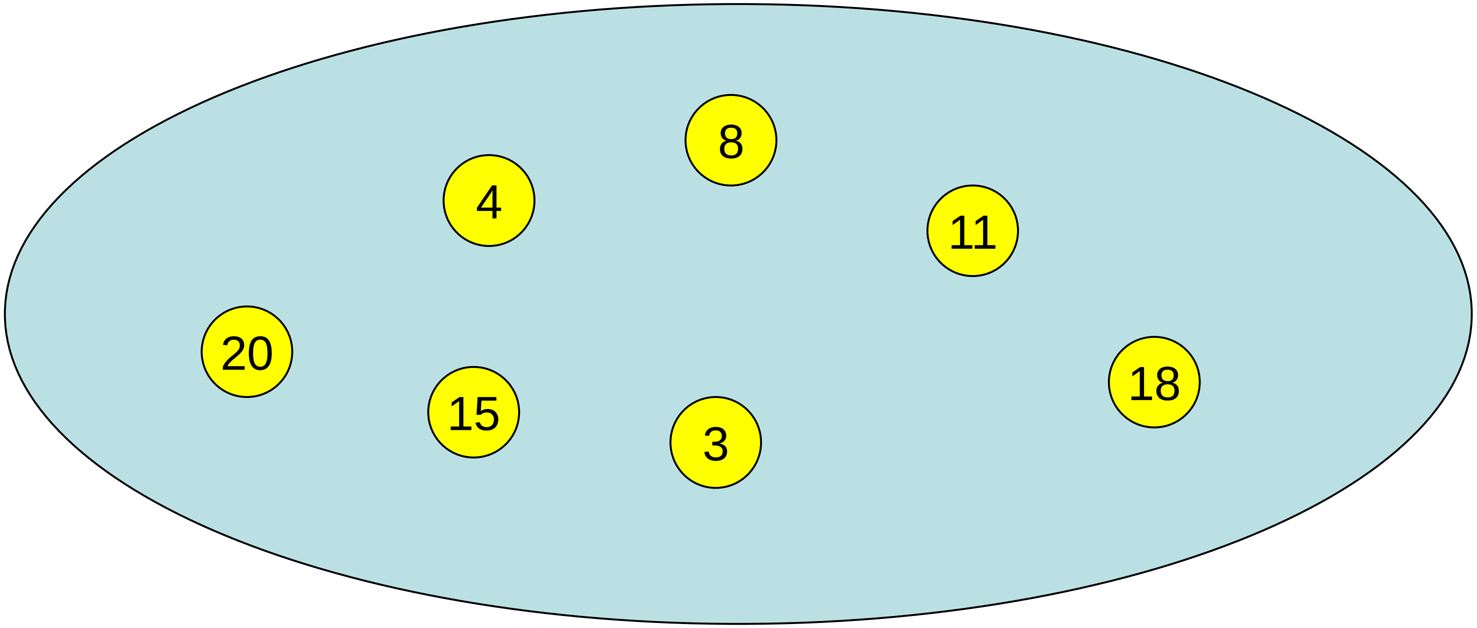
Search Structure

insert(15)



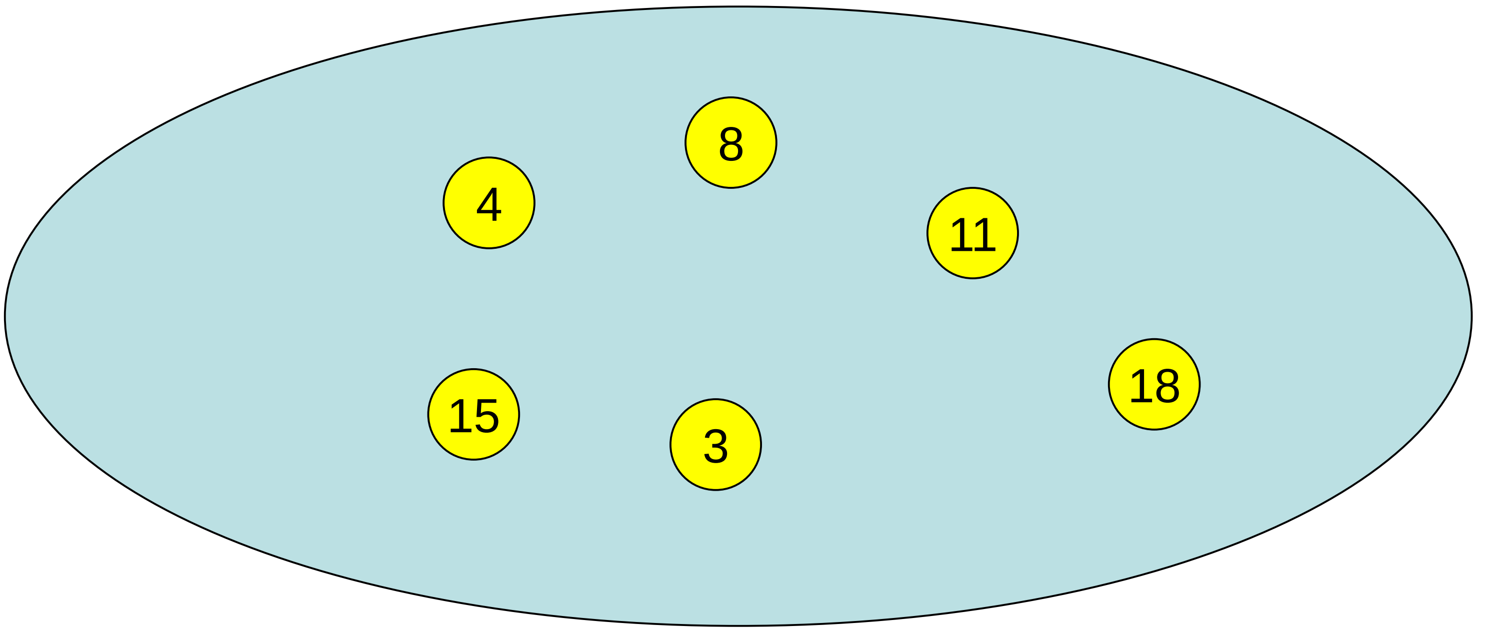
Search Structure

delete(20)



Search Structure

search(7) gives 8 (closest successor)



Search Structure

S: set of elements

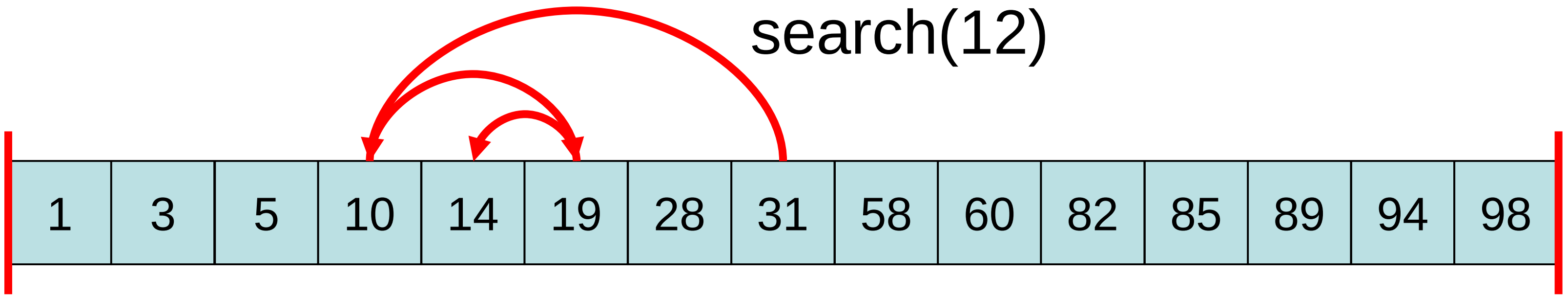
Every element e identified by key(e).

Operations:

- S.insert(e: Element): $S:=S\cup\{e\}$
- S.delete(k: Key): $S:=S\setminus\{e\}$, where e is the element with key(e)=k
- S.search(k: Key): outputs $e\in S$ with minimal key(e) so that $key(e)\geq k$

Static Search Structure

1. Store elements in sorted array.



search: via binary search (in O(log n) time)

Binary Search

Input: number x and sorted array A[1],…,A[n]

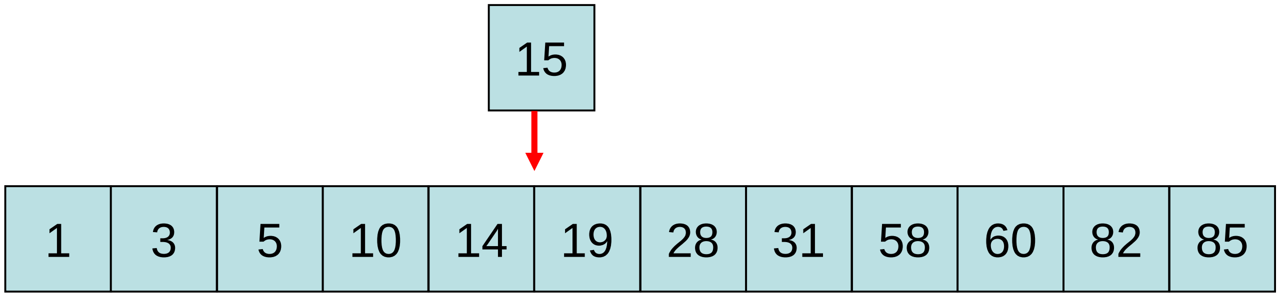
Algorithm BinarySearch:

```
l:=1; r:=n
while l < r do
    m:=(r+l) div 2
    if A[m] = x then return m
    if A[m] < x then l:=m+1
                 else r:=m
return l
```

Dynamic Search Structure

insert und delete Operations:

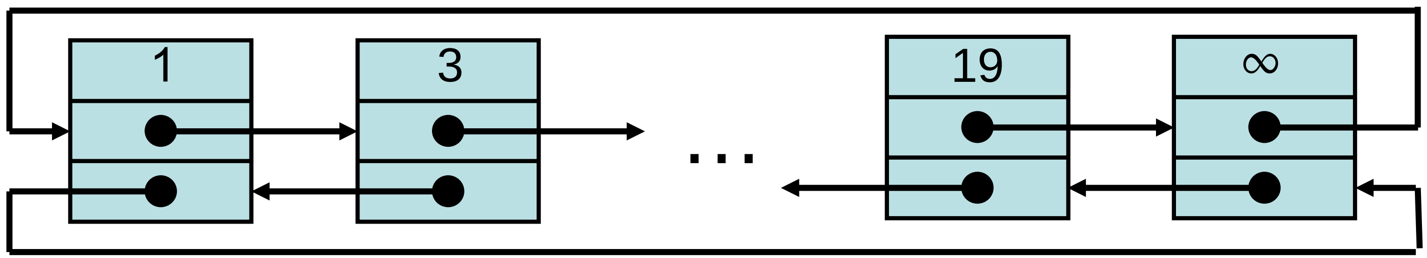
Sorted array difficult to update!



Worst case: $\Theta(n)$ time

Search Structure

2. Sorted List (with an ∞-Element)

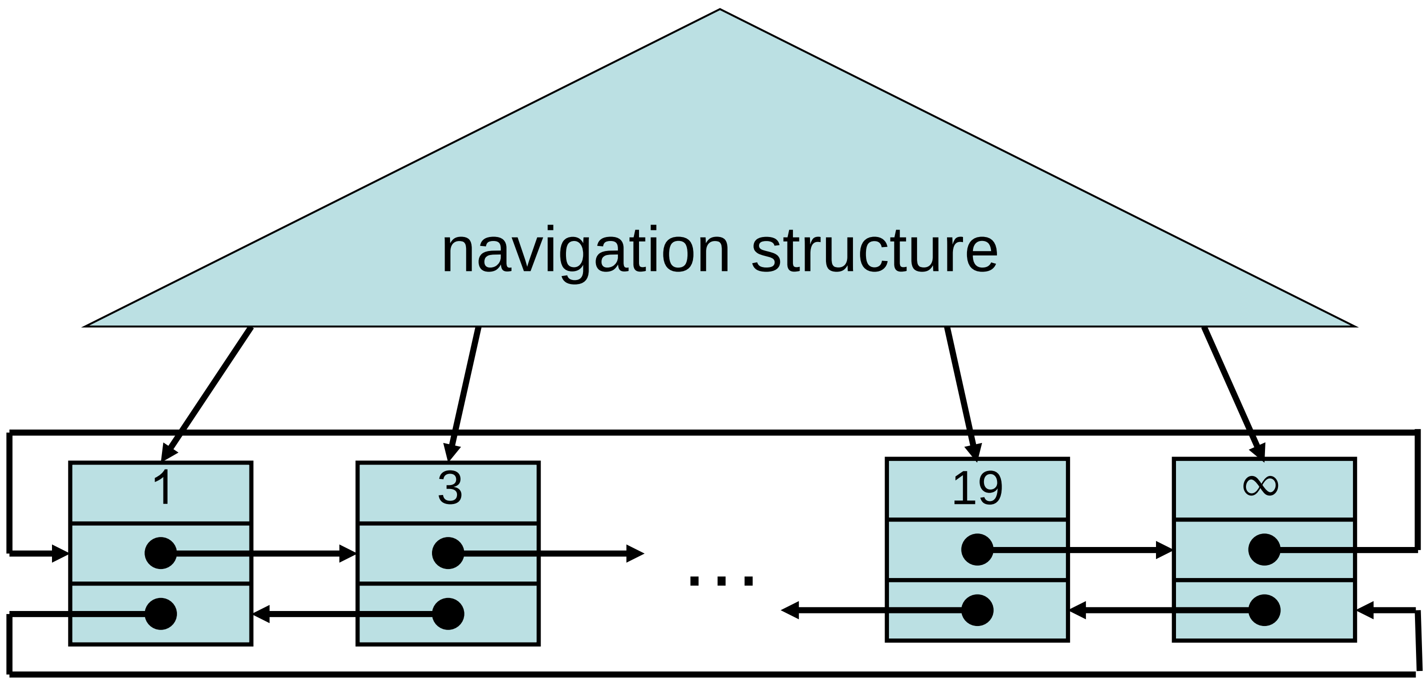


Problem: insert, delete and search take $\Theta(n)$ time in the worst case

Observation: If search could be implemented efficiently, then also all other operations

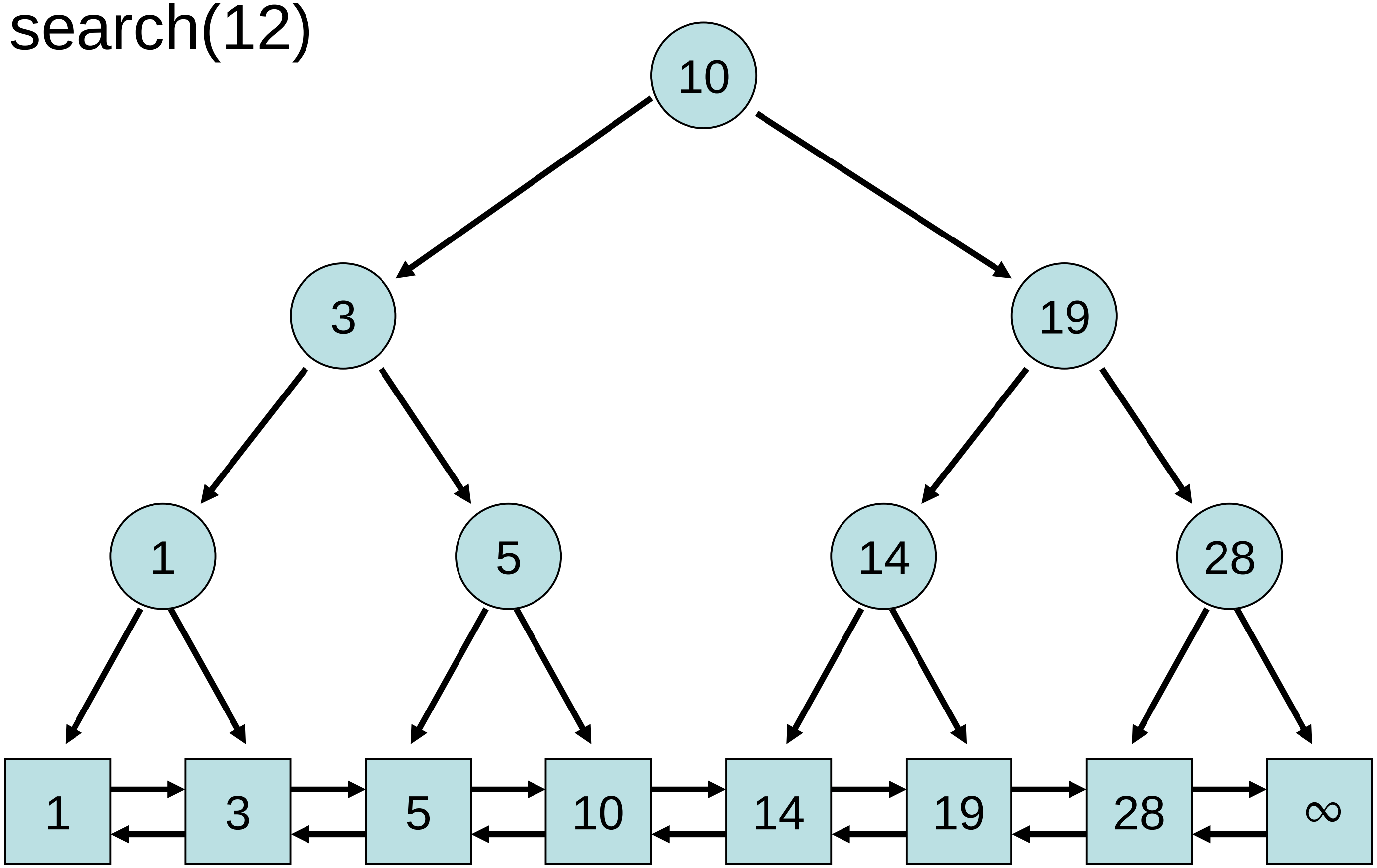
Search Structure

Idee: add navigation structure that allows search to run efficiently



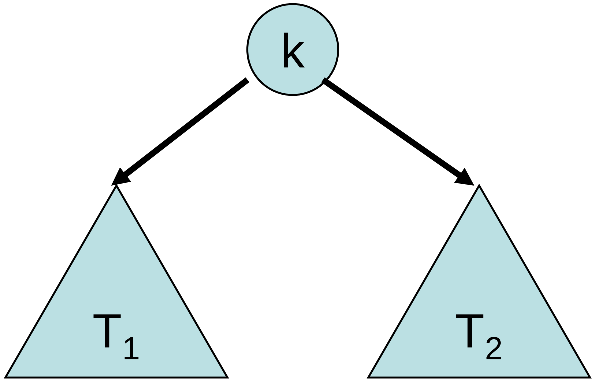
Binary Search Tree (ideal)

search(12)



Binary Search Tree

Search tree invariant:



For all keys k' in T_1 and k'' in T_2: $k' \leq k < k''$

Binary Search Tree

Formally: for every tree node v let

- key(v) be the key stored at v
- d(v) the number of children of v

- Search tree invariant: (as above)

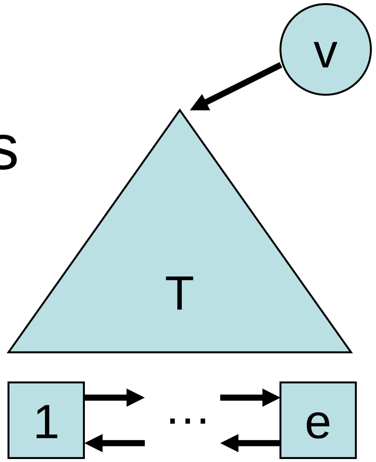
- Degree invariant:
  All tree nodes have exactly two children
  (as long as the number of elements in the list is >1)

- Key invariant:
  For every element e in the list there is exactly one tree node v with key(v)=key(e).

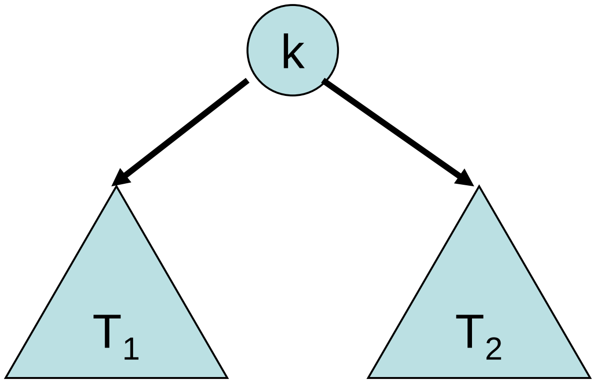
Binary Search Tree

- Search tree invariant: (as before)
- Degree invariant:
  All tree nodes have exactly two children
  (as long as the number of elements is >1)
- Key invariant:
  For every element e in the list there is exactly one tree node v with key(v)=key(e).

From the search tree and key invariants it follows that for every left subtree T of a node v, the rightmost list element e under T satisfies key(v)=key(e).



search(x) Operation

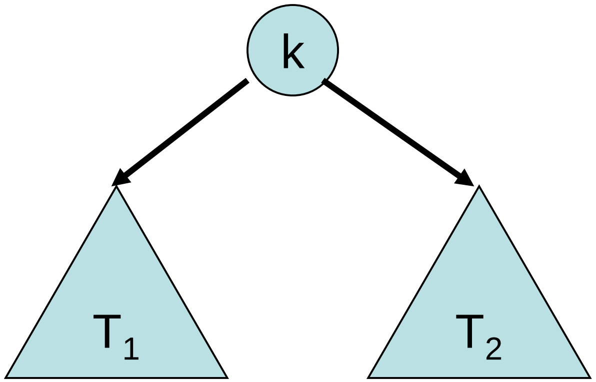


For all keys k' in T_1 and k'' in T_2: $k' \leq k < k''$

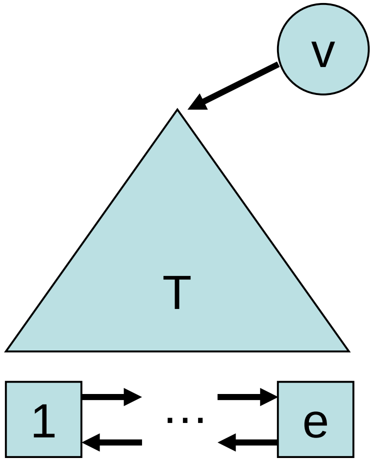
Search strategy:

- Start at the root, v, of the search tree
- while v is a tree node:
  – if $x \leq key(v)$ then let v be the left child of v, otherwise let v be the right child of v
- Output (list node) v

search(x) Operation



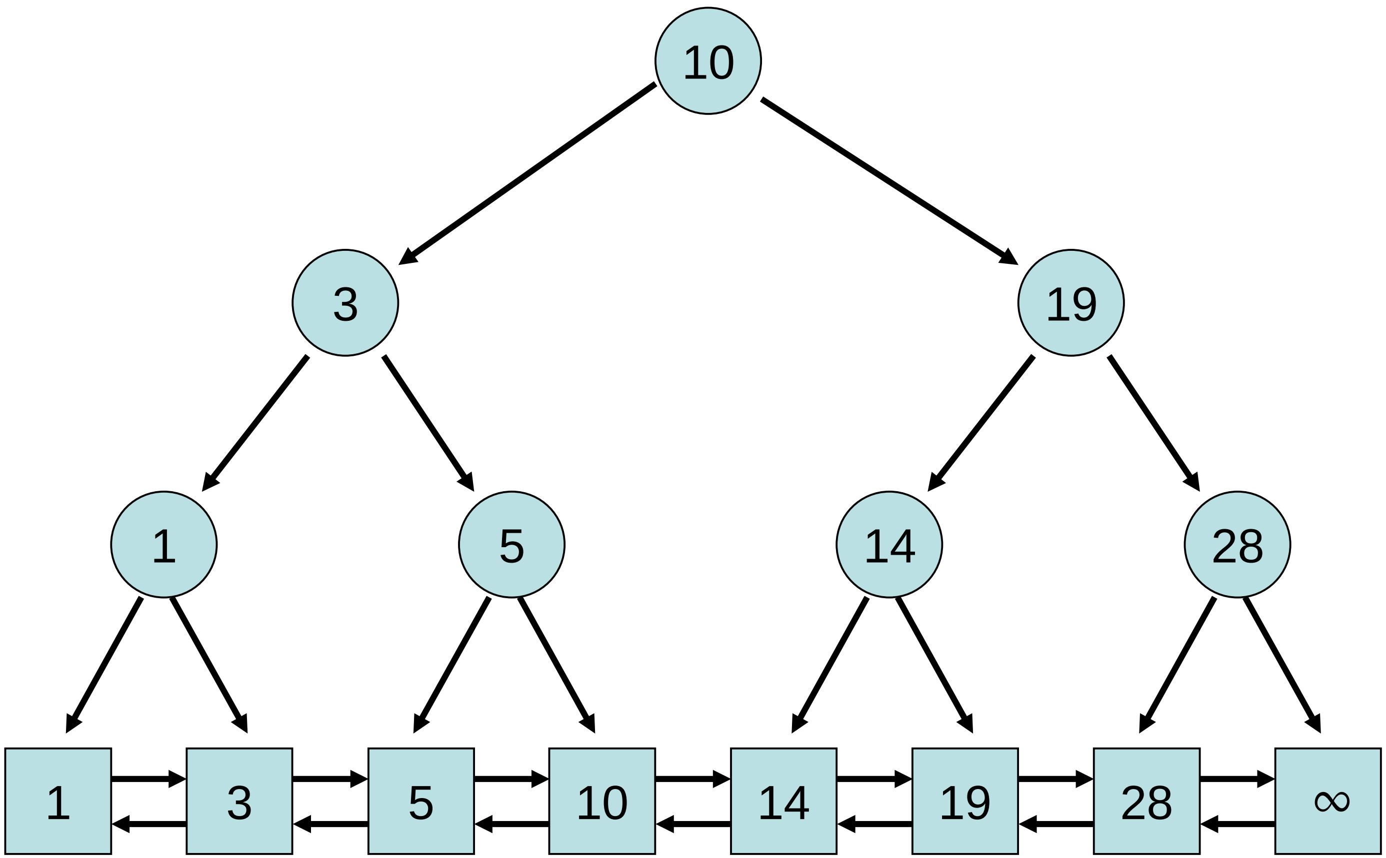
For all keys k´ in T_1 and k´´ in T_2: k´ ≤ k < k´´



Correctness of search strategy:

- For every left subtree T of a node v, the rightmost list element e under T satisfies key(v)=key(e).
- So whenever search(x) enters T, there is an element e in the list below T with key(e)≥x.

Search(9)



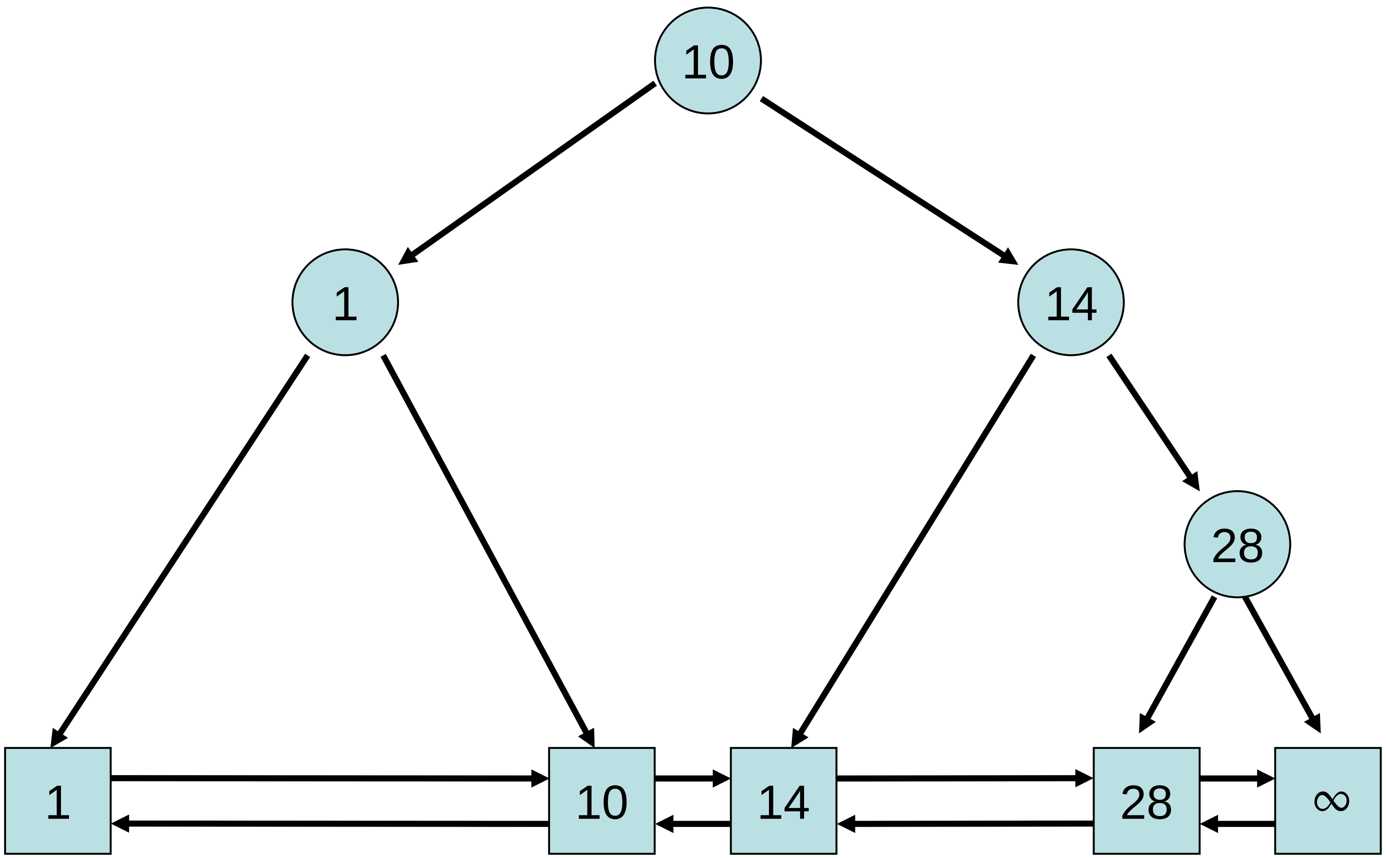
Insert and Delete Operations

Strategy:

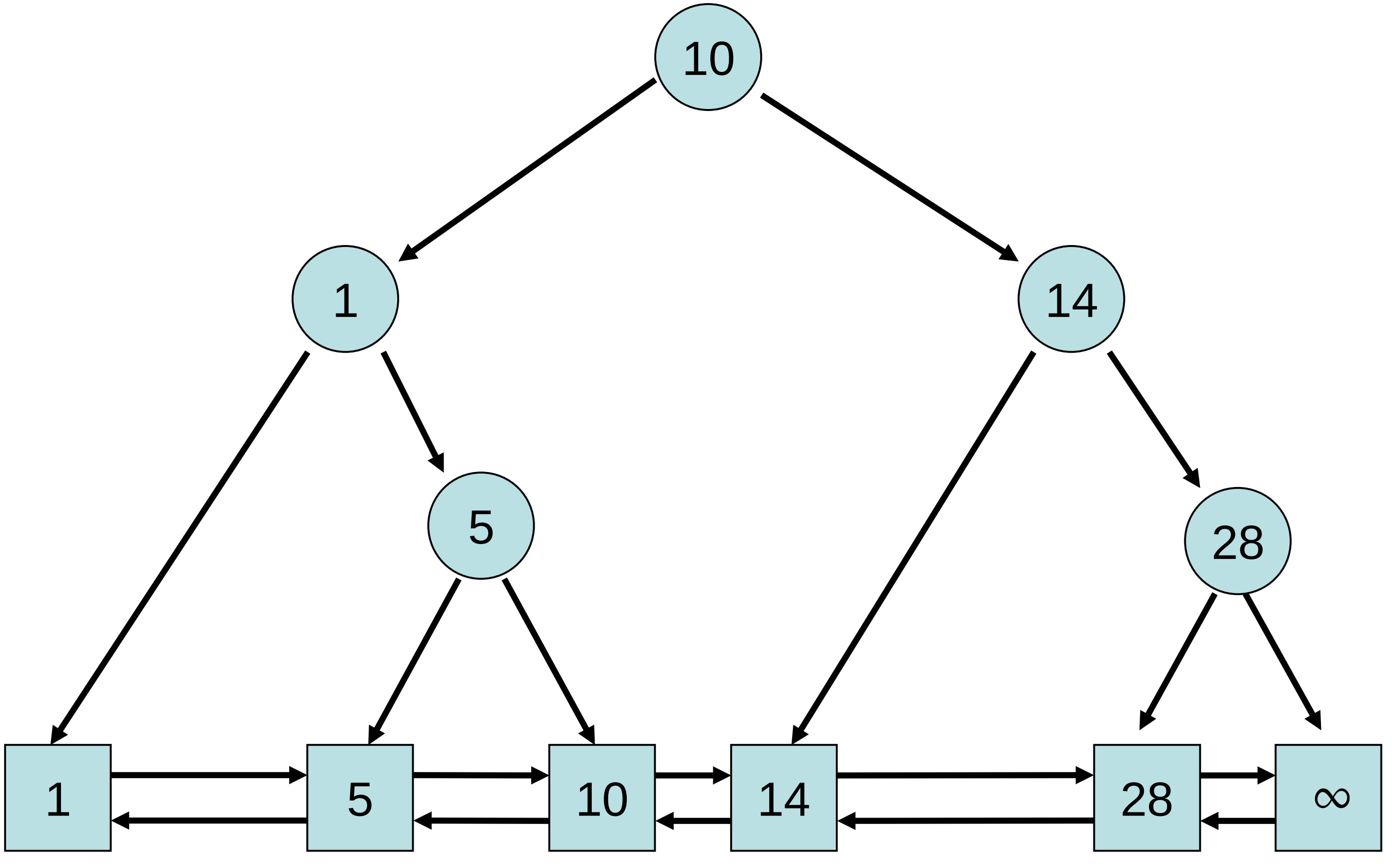
- insert(e):
  First, execute search(key(e)) to obtain a list element e´. If key(e)=key(e´), replace e´ by e, otherwise insert e between e´ and its predecessor in the list and add a new search tree leaf for e and e´ with key key(e).

- delete(k):
  First, execute search(k) to obtain a list element e. If key(e)=k, then delete e from the list and the parent v of e from the search tree, and set in the tree node w with key(w)=k: key(w):=key(v).

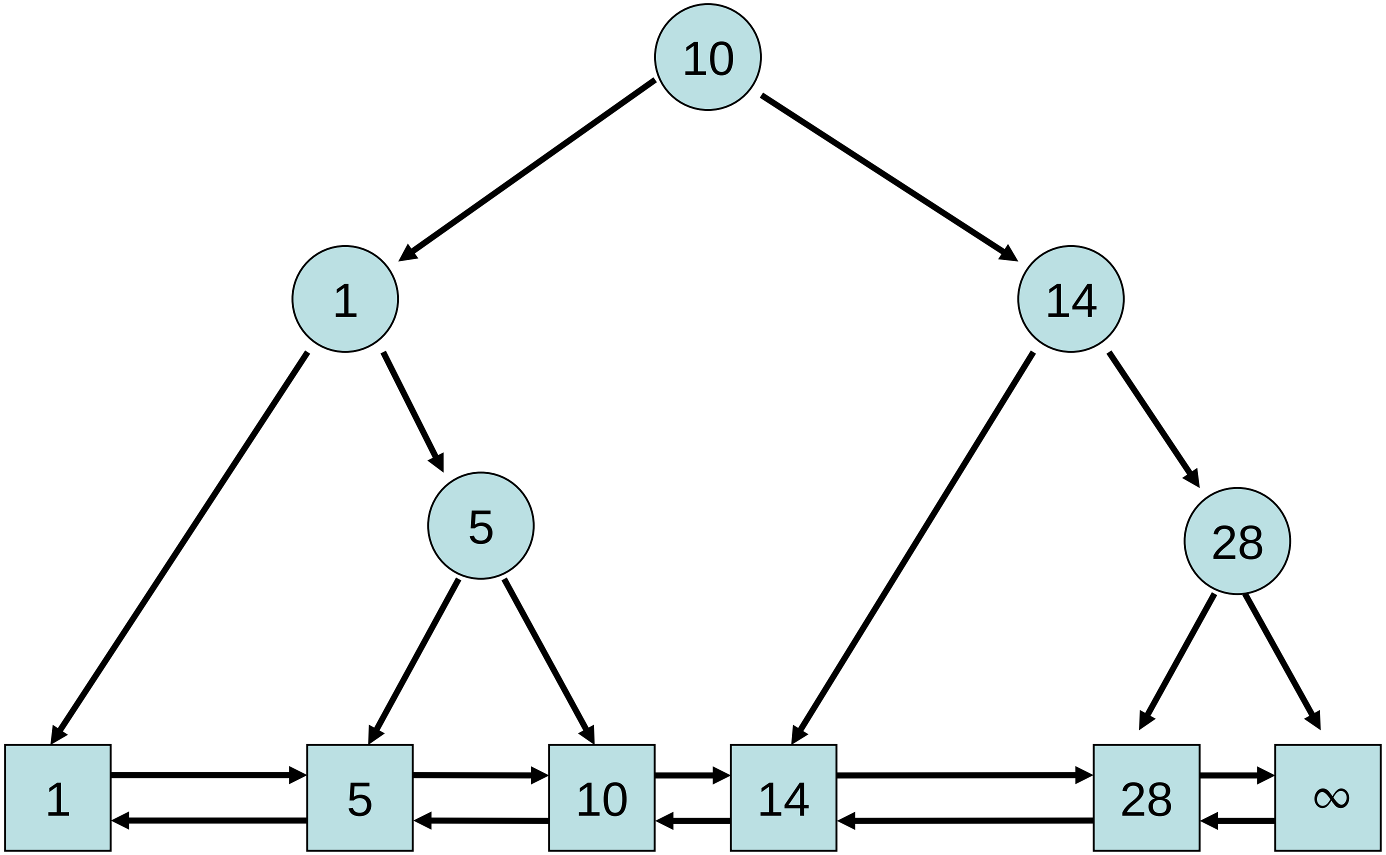
Insert(5)



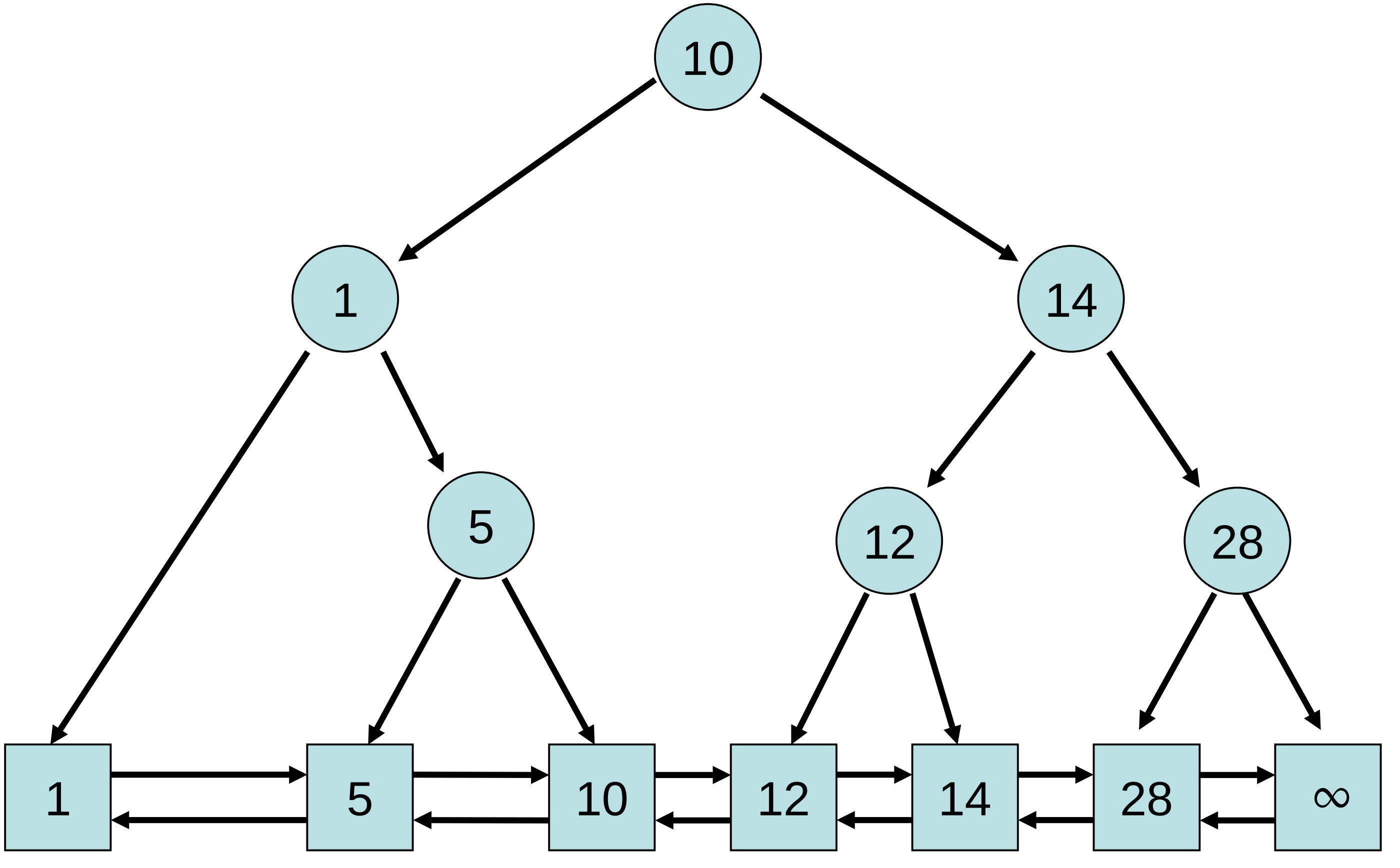
Insert(5)



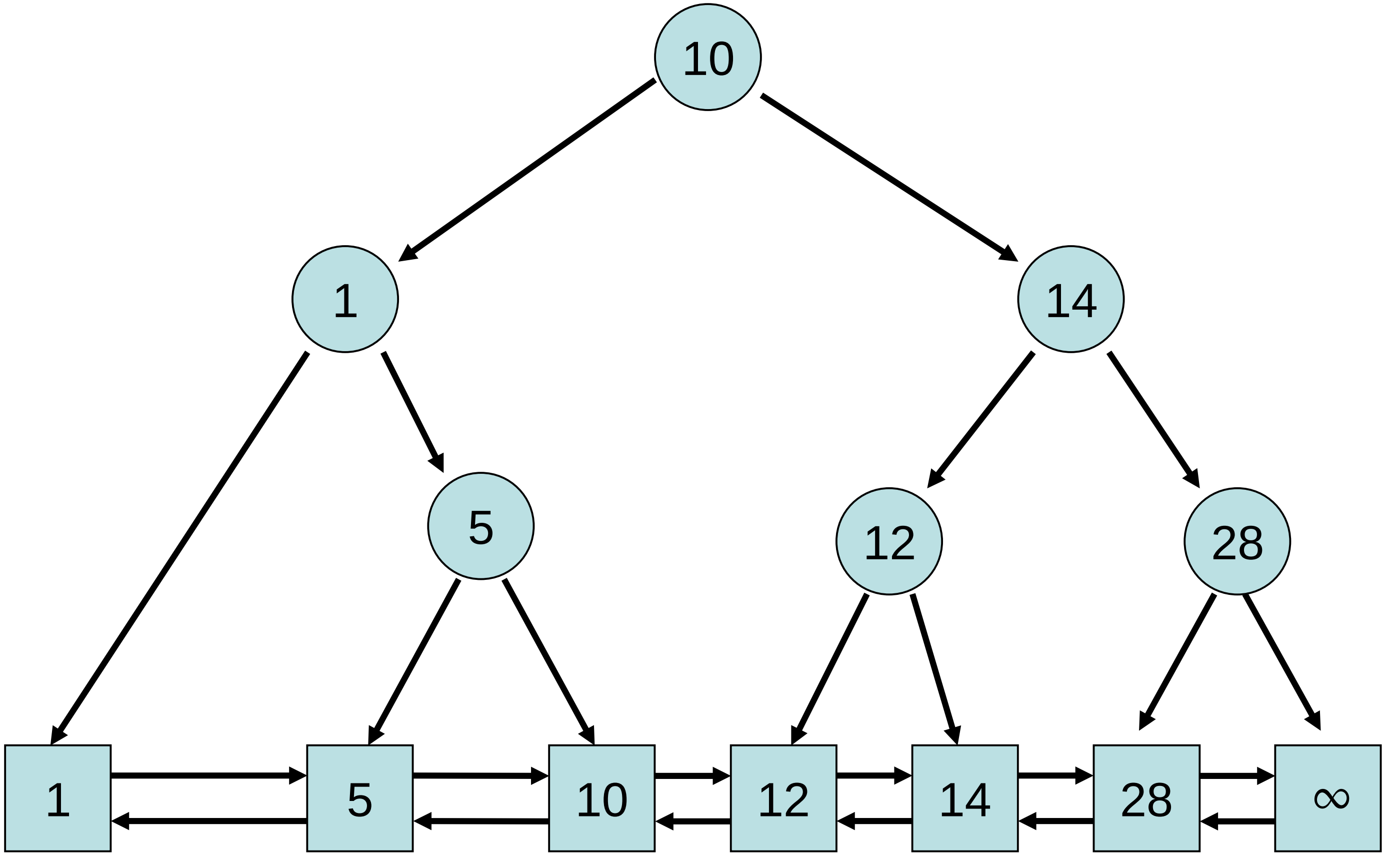
Insert(12)



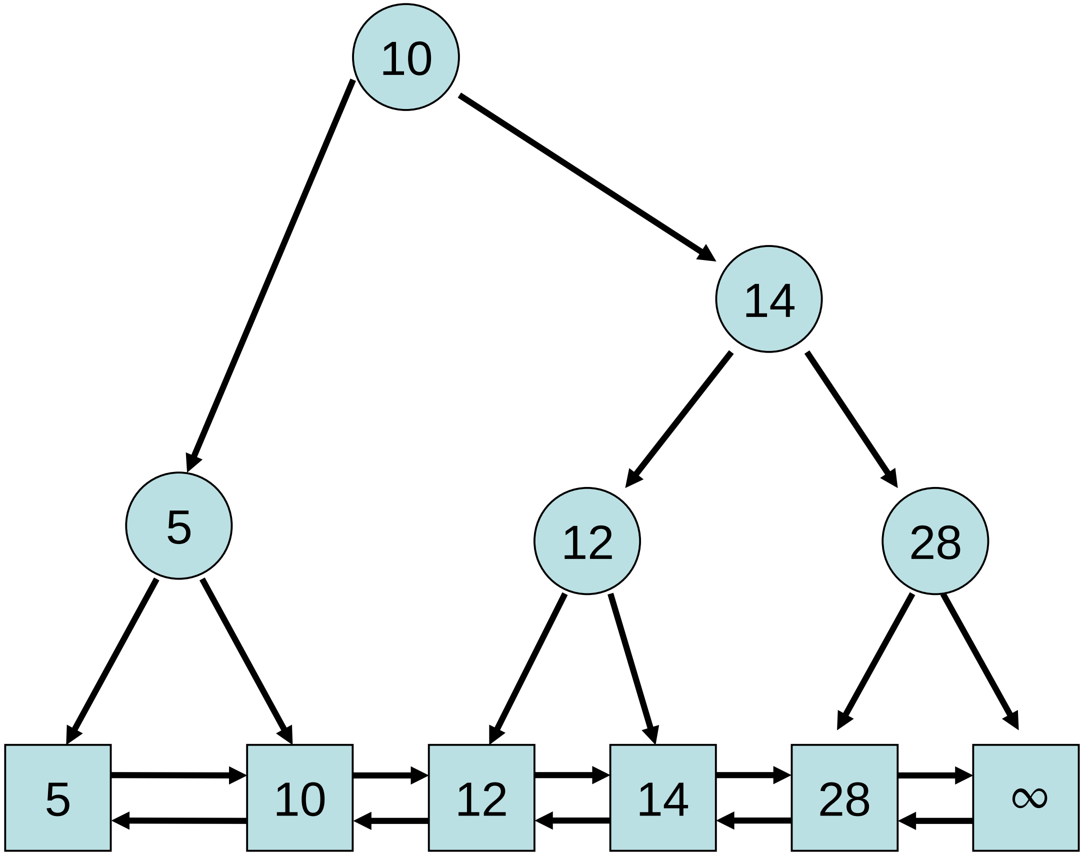
Insert(12)



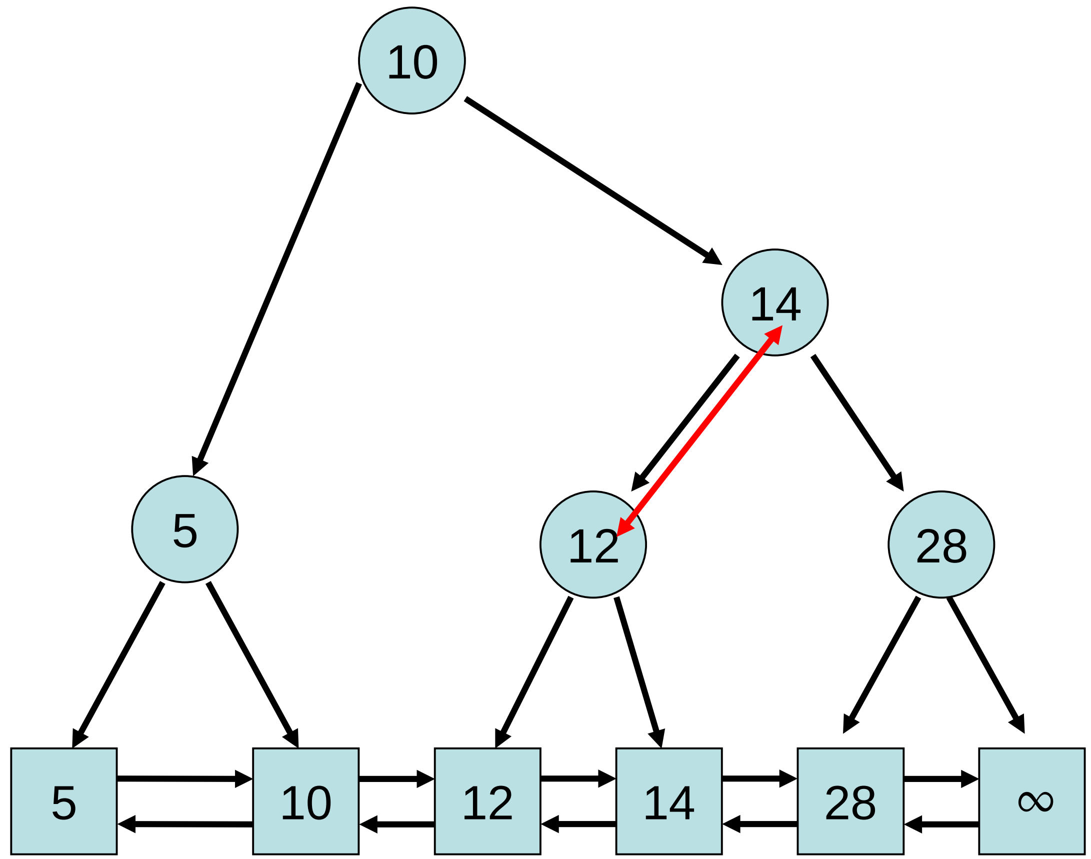
Delete(1)



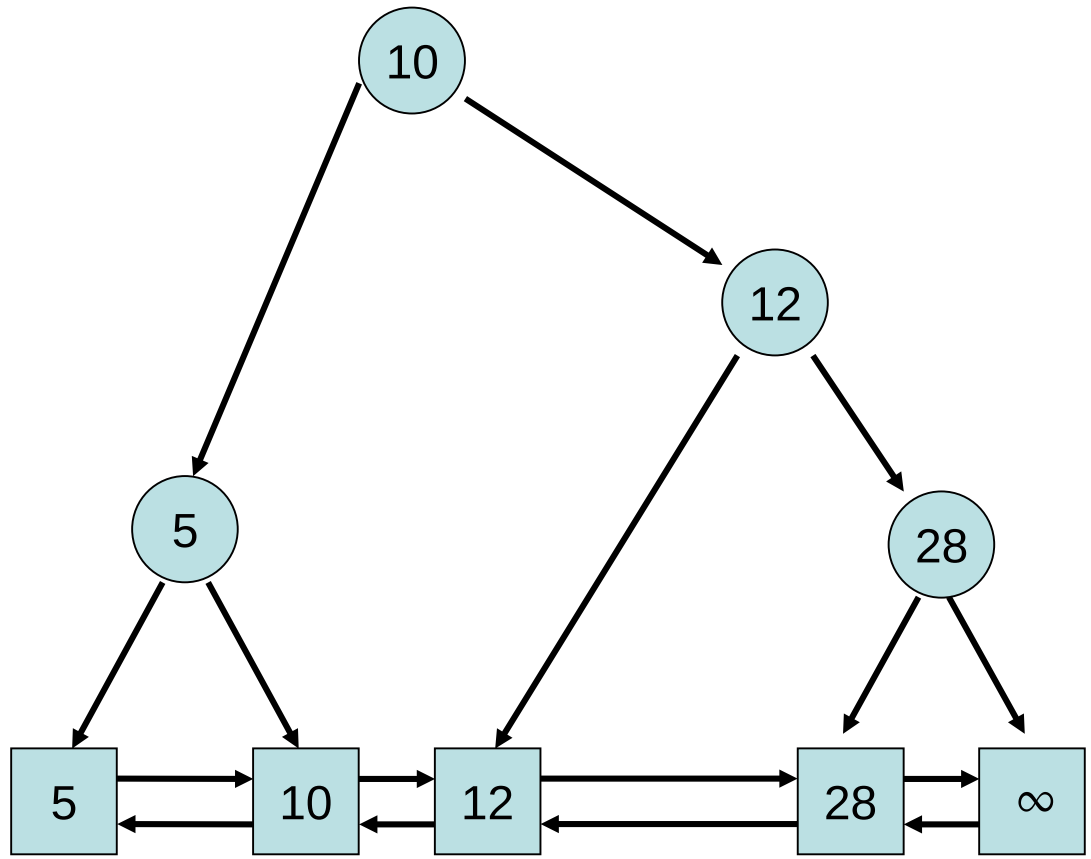
Delete(1)



Delete(14)



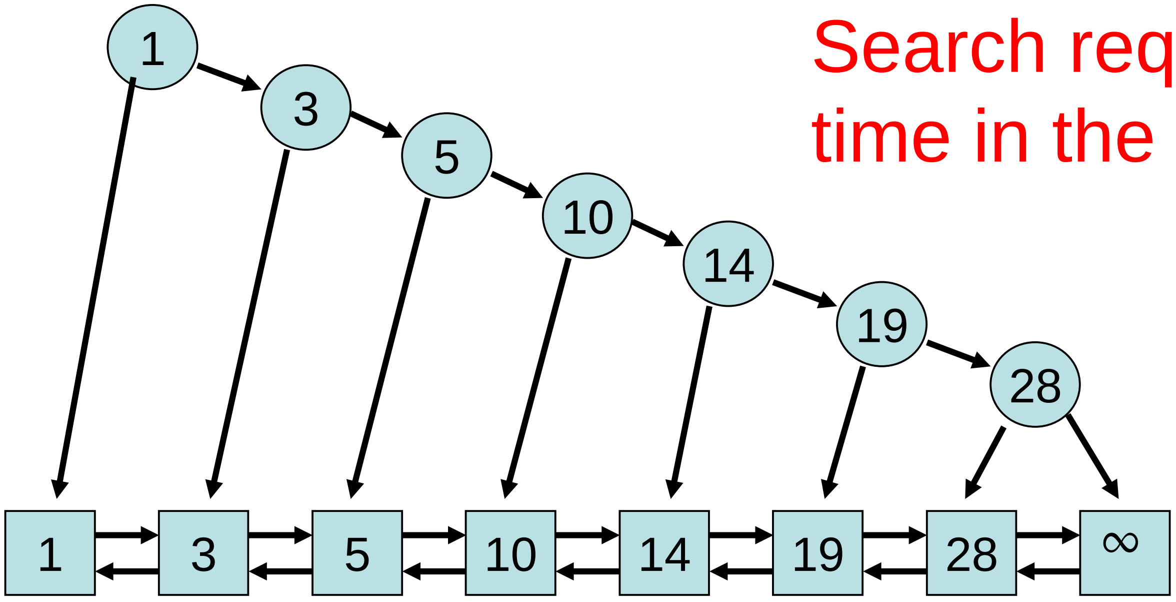
Delete(14)



Binary Search Tree

Problem: binary tree can degenerate!

Example: numbers are inserted in sorted order



Search requires Θ(n) time in the worst case

Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- hashed Patricia trie
  (loglog-search time)

Applications

Splay Tree

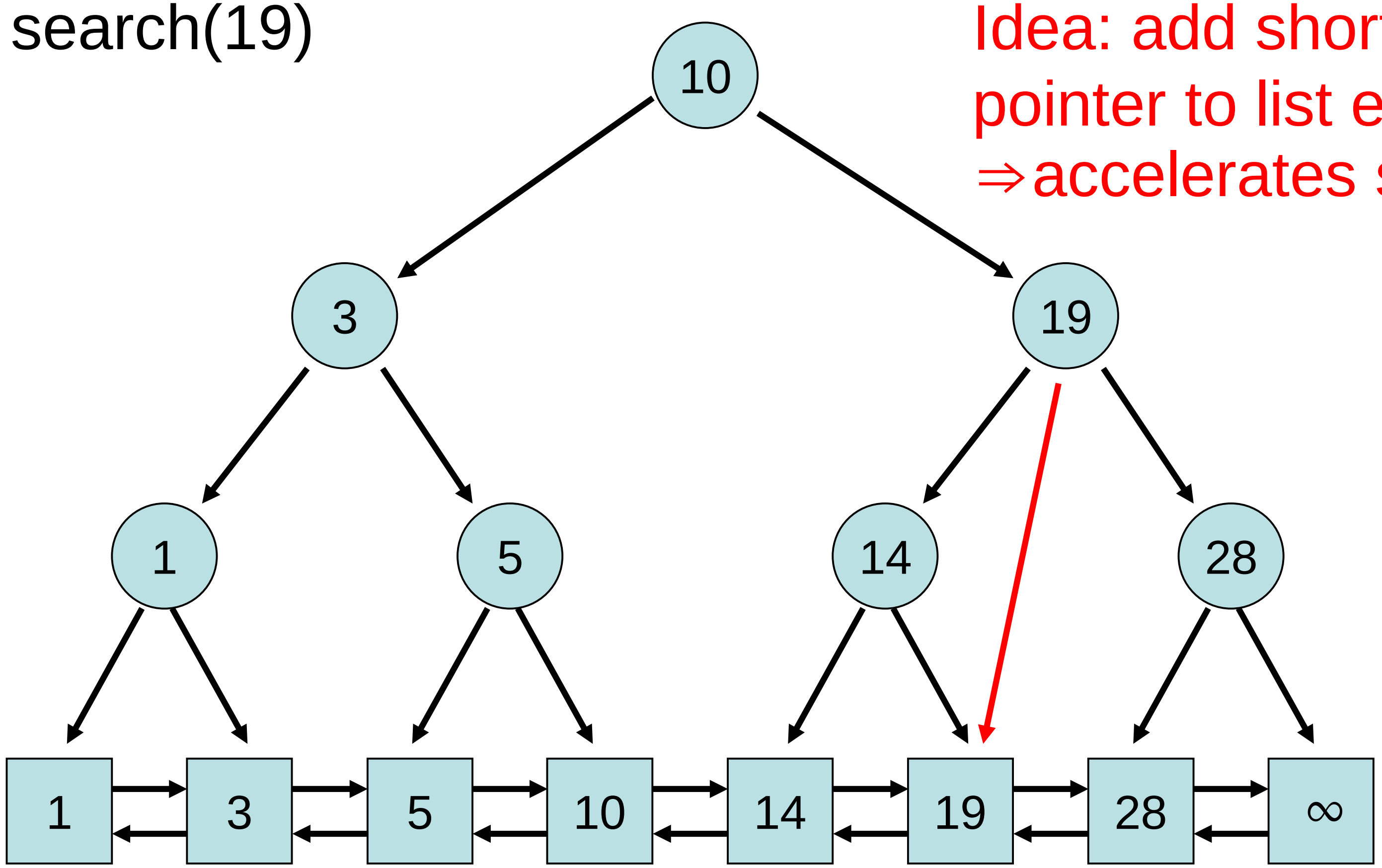
Usually: Implementation as internal search tree (i.e., elements directly integrated into tree and not in an extra list)

Here: Implementation as external search tree (like for the binary search tree above)

Splay Tree

search(19)

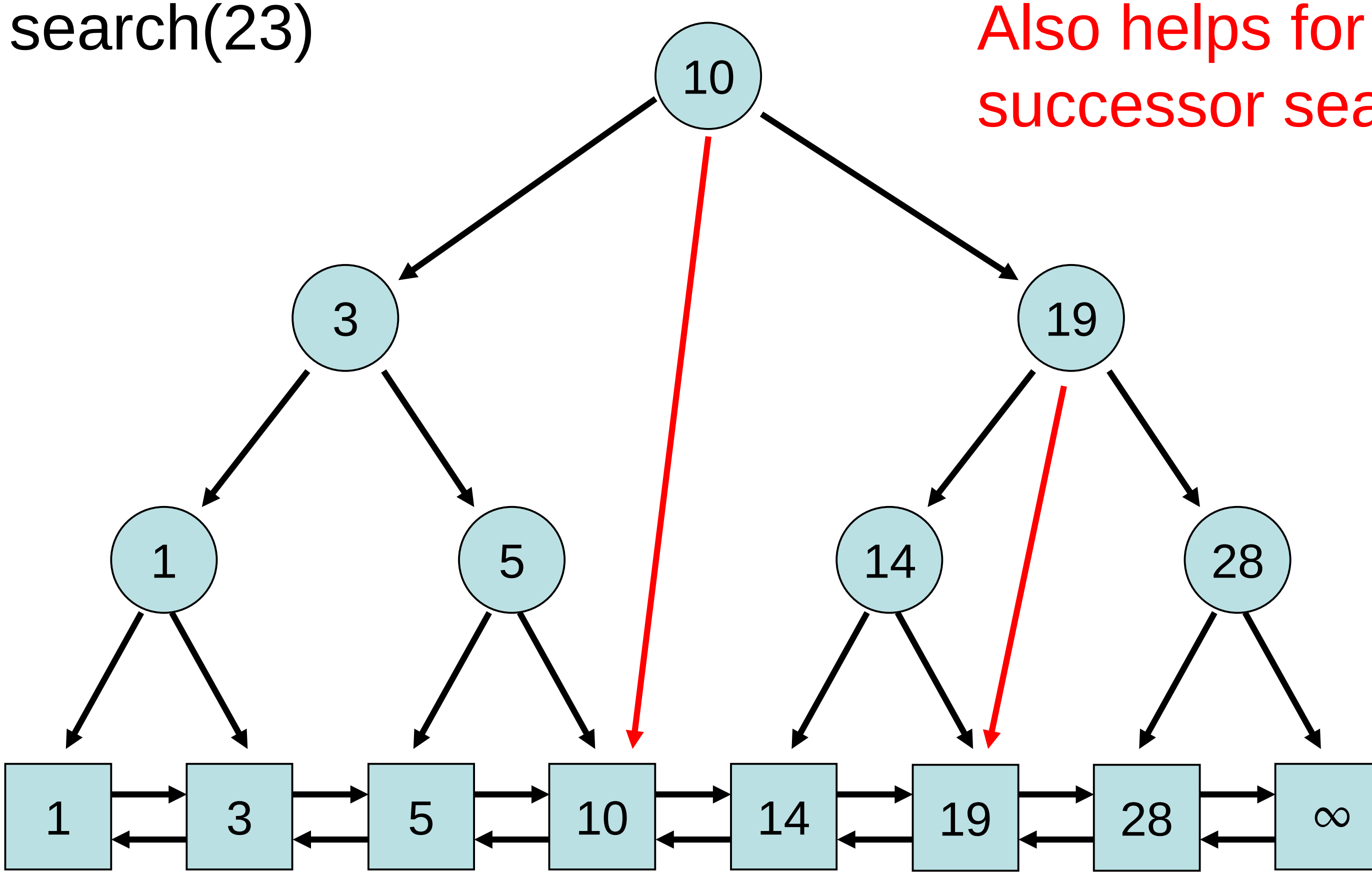
Idea: add shortcut pointer to list element
⇒ accelerates search



Splay Tree

search(23)

Also helps for closest successor search



Splay Tree

Ideas:

1. Add shortcut pointers in tree to list elements
2. For every search(k) operation, move pred(k) (the closest predecessor of k in T) to the root

Movement: via Splay operation

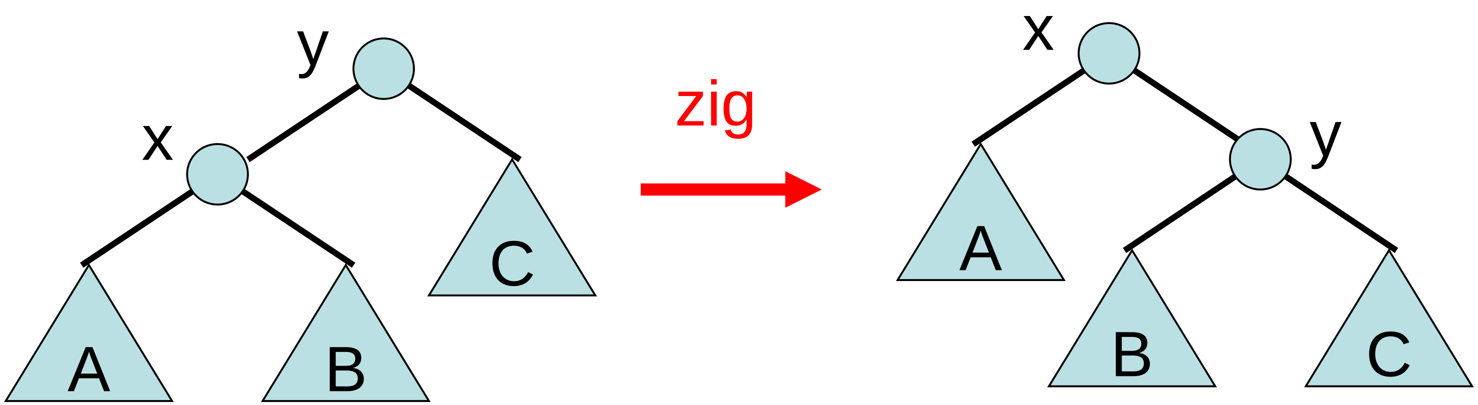
For simplicity: we just focus on search(k) for keys k in the search tree.

Splay Operation

Movement of key x to the root:

We distinguish between three cases.

1a. x is a child of the root:

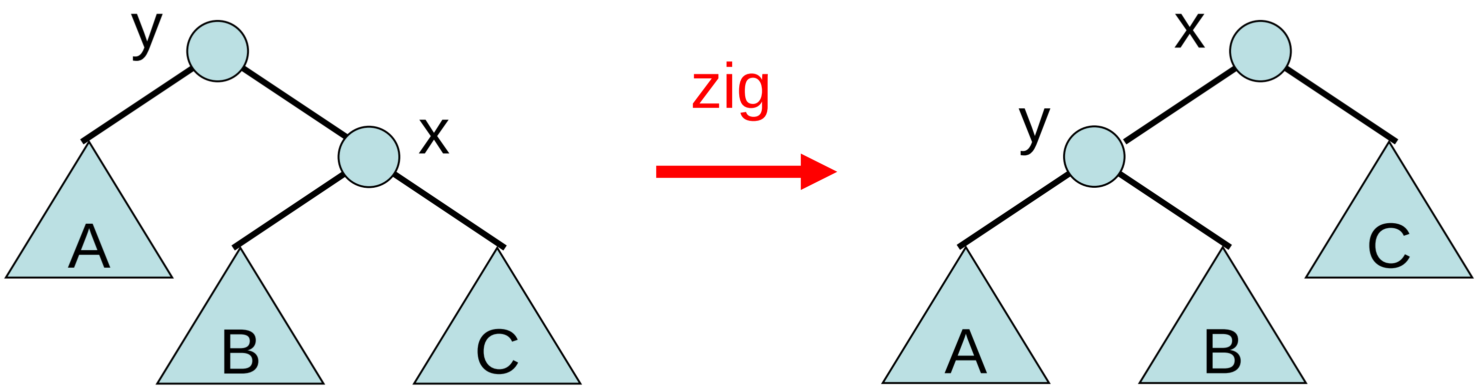


Splay Operation

Movement of key x to the root:

We distinguish between three cases.

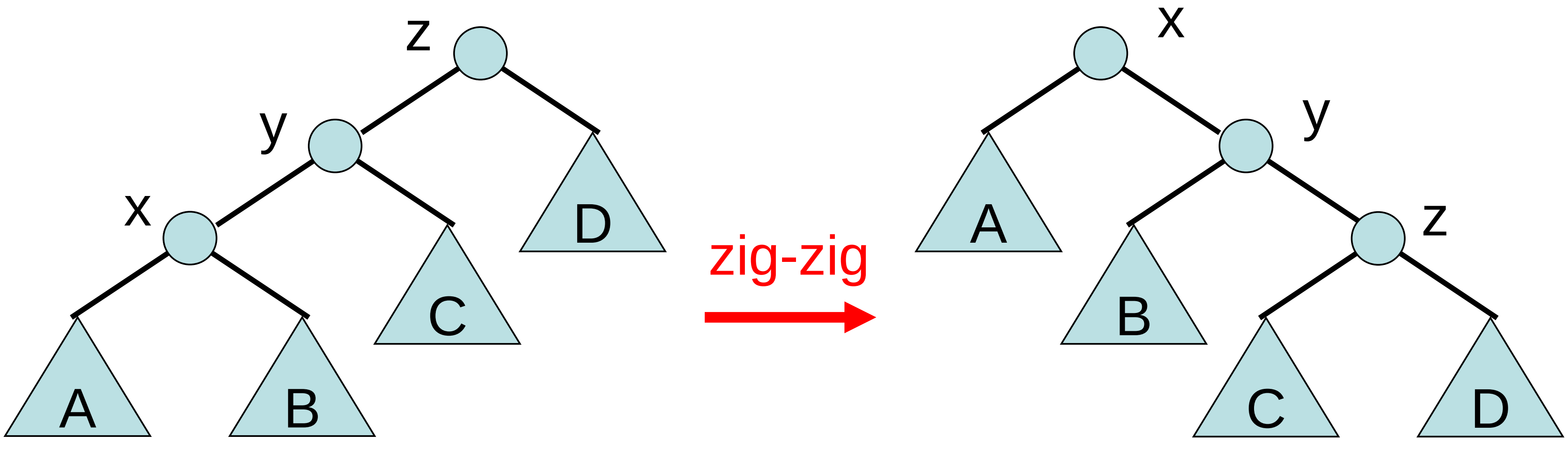
1b. x is a child of the root:



Splay Operation

We distinguish between three cases.

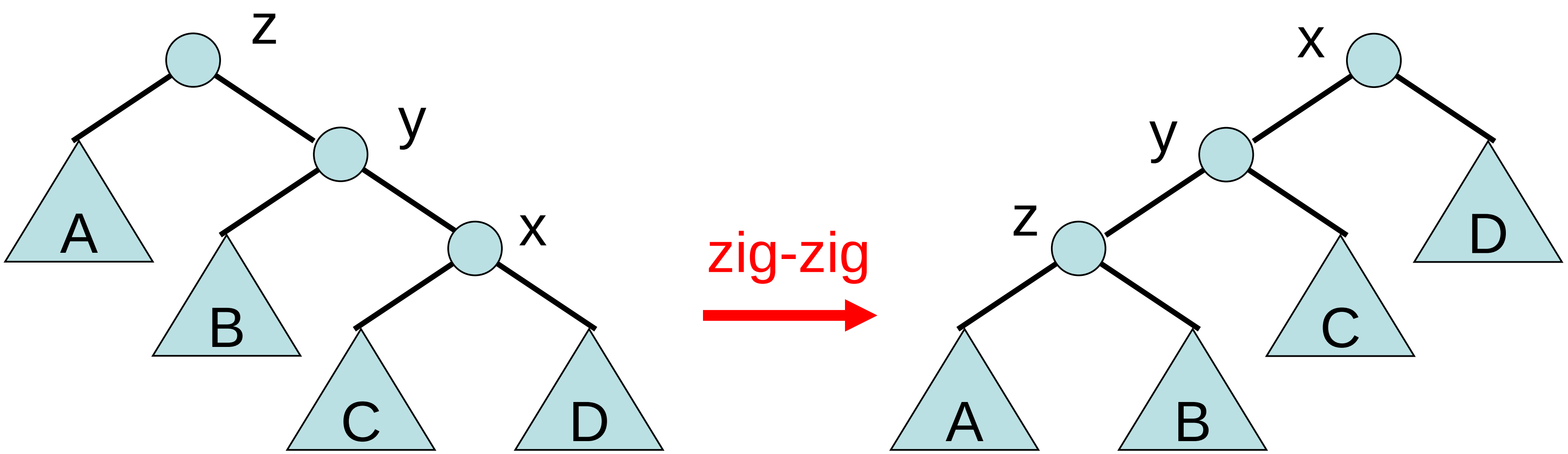
2a. x has father and grand father to the right



Splay Operation

We distinguish between three cases.

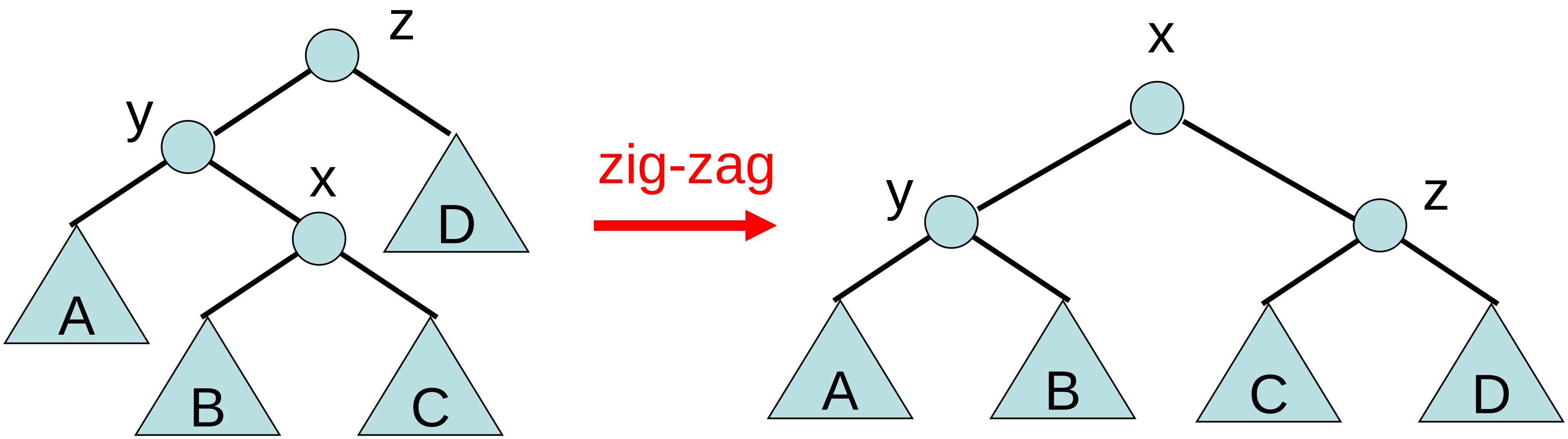
2b. x has father and grand father to the left



Splay Operation

We distinguish between three cases.

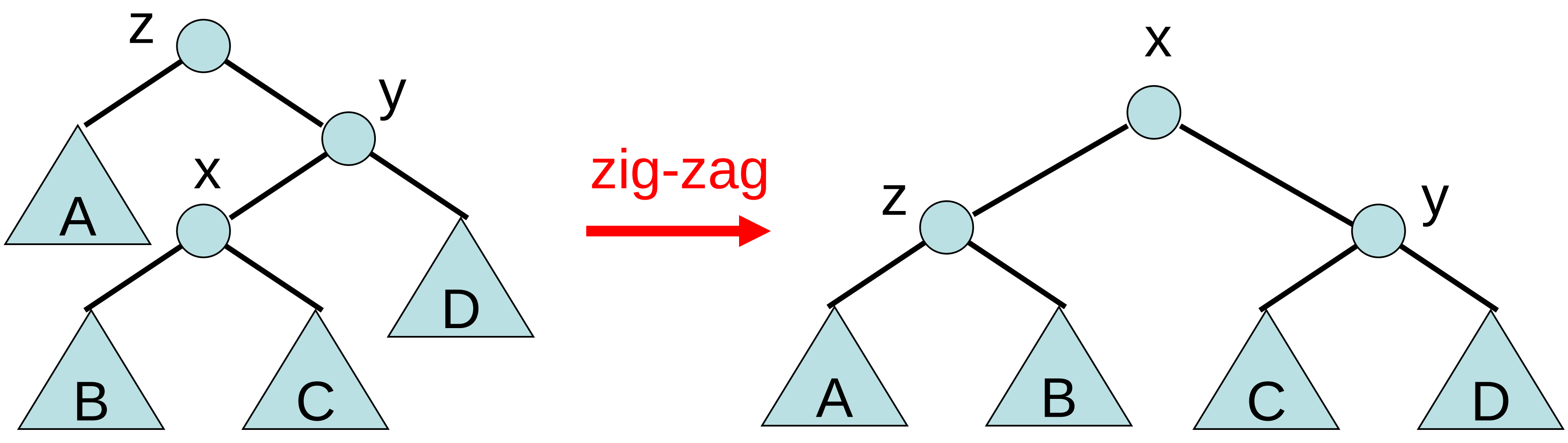
3a. x: father left, grand father right



Splay Operation

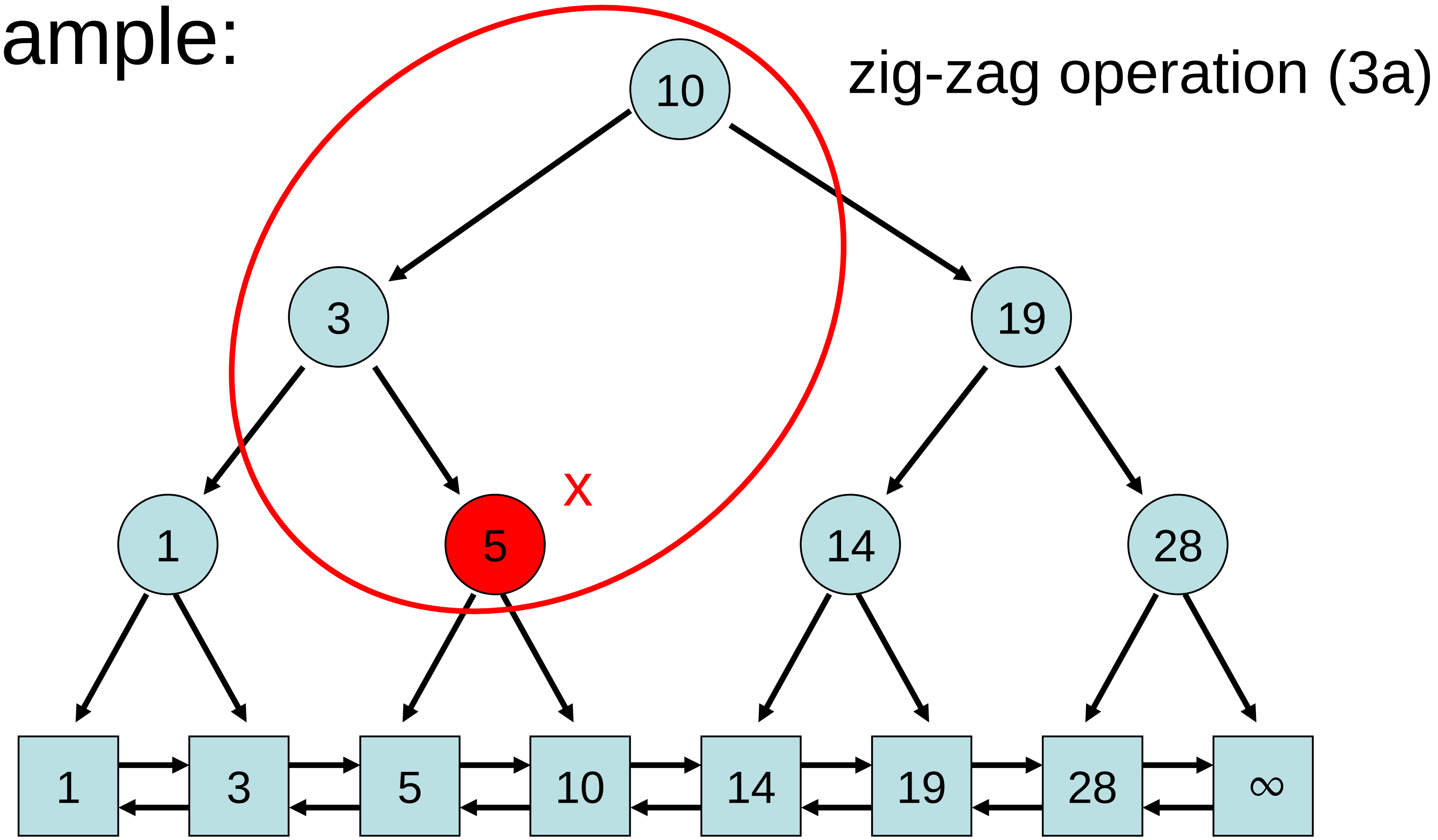
We distinguish between three cases.

3b. x: father right, grand father left

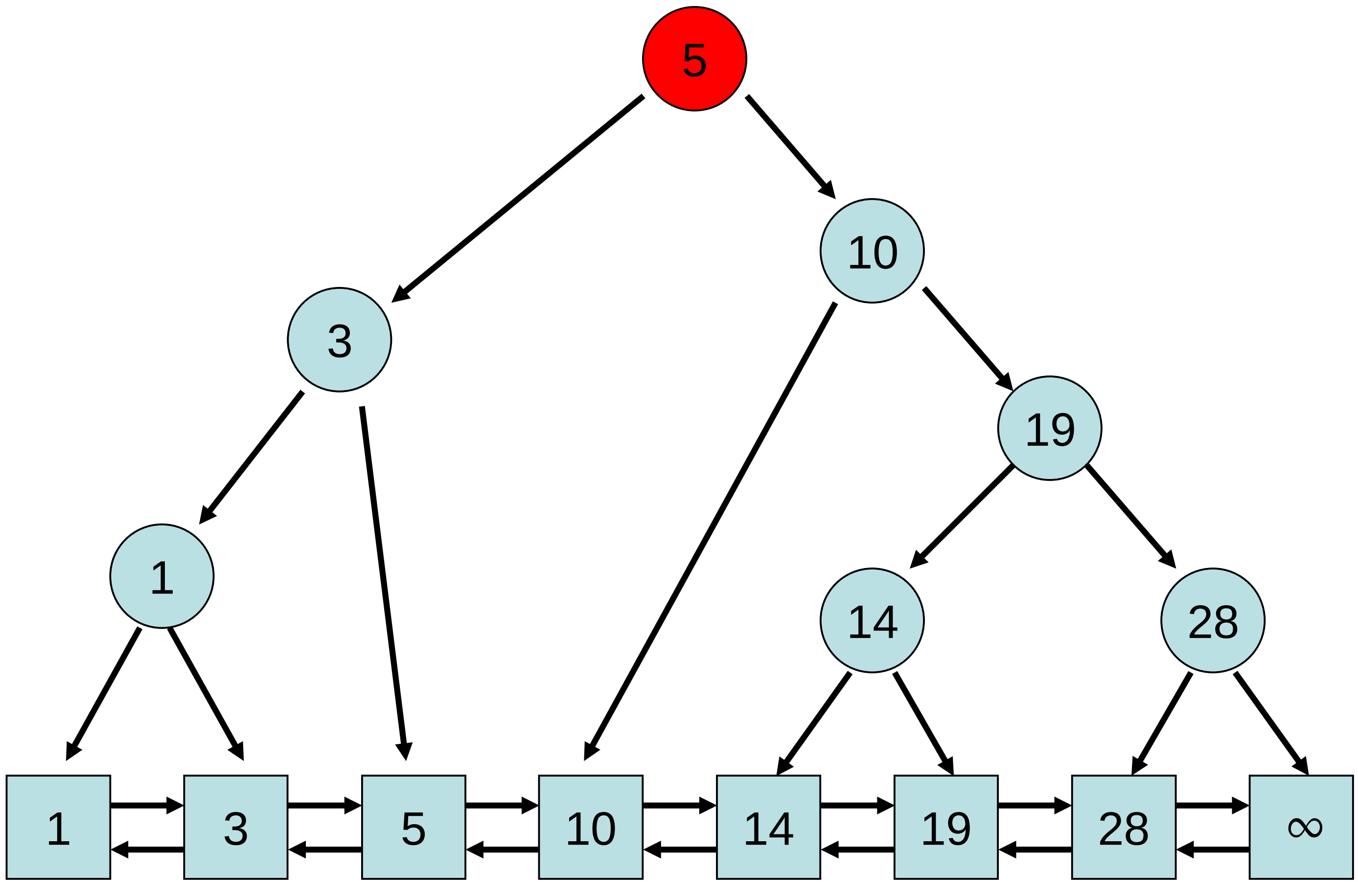


Splay Operation

Example:

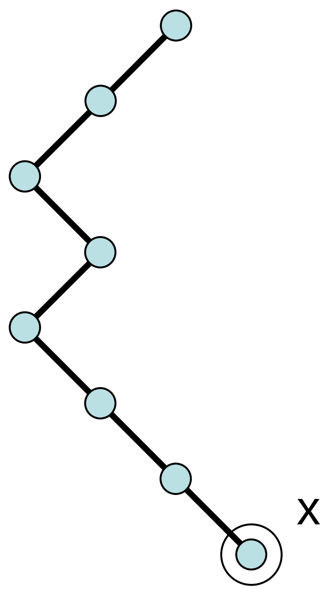


Splay Operation

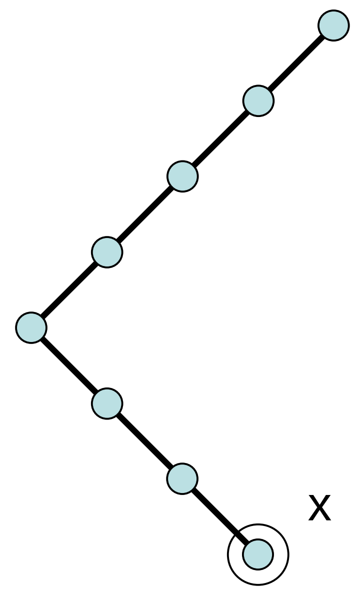


Splay Operation

Examples:



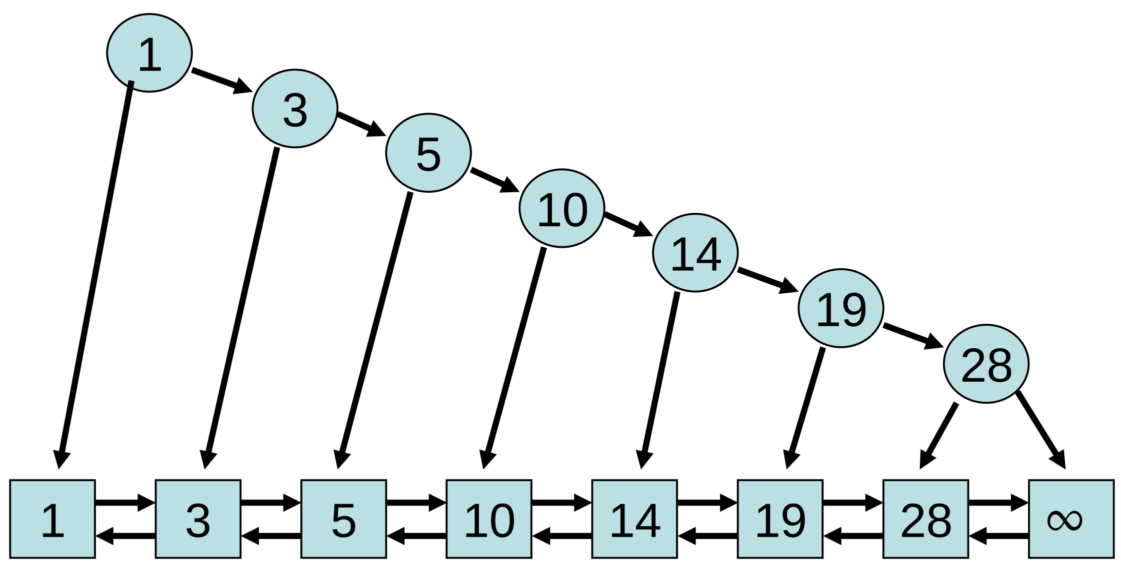
zig-zig, zig-zag, zig-zag, zig



zig-zig, zig-zag, zig-zig, zig

Splay Operation

Observation: In the worst case, tree can still be highly imbalanced! But amortized costs are very low.



Splay Operation

search(k)-operation:

- Move downwards from the root (as in standard binary tree) till pred(k) found in search tree (which can be checked via shortcut to the list) or the list is reached
- call splay(pred(k)), output succ(k) (k exists in tree: pred(k)=succ(k)=k)

Amortized Analysis:

- Note: runtime of search(k) is O(runtime of splay(pred(k))).
- Our goal: bound runtime of m Splay operations on arbitrary binary search tree with n elements (m>n)

Splay Operation

- Weight of node x: $w(x)>0$
- Tree weight of tree T with root x: $tw(x)= \sum_{y \in T} w(y)$
- Rank of node x: $r(x) = \log(tw(x))$
- Potential of tree T: $\phi(T) = \sum_{x \in T} r(x)$

Lemma 3.1: Let T be a Splay tree with root x and u be a node in T. The amortized cost for splay(u,T) is at most $1+3(r(x)-r(u))$.

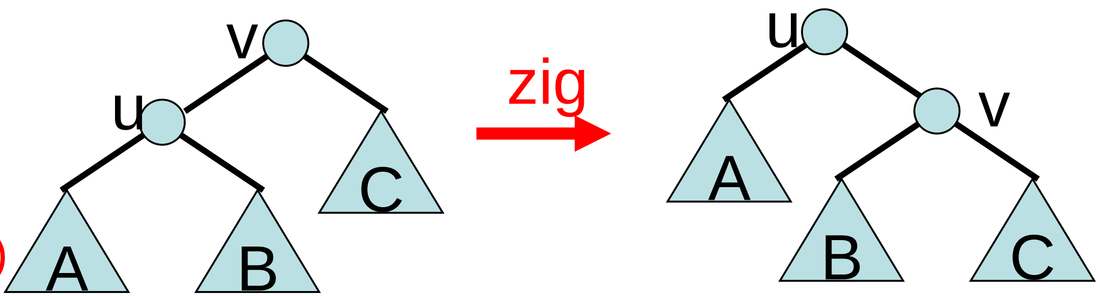
Splay Operation

Proof of Lemma 3.1:

Induction over the sequence of rotations.

- r and tw : rank and weight before the rotation
- r’ and tw’: rank and weight after the rotation

Case 1:



Runtime
(# rotations)

Amortized cost:

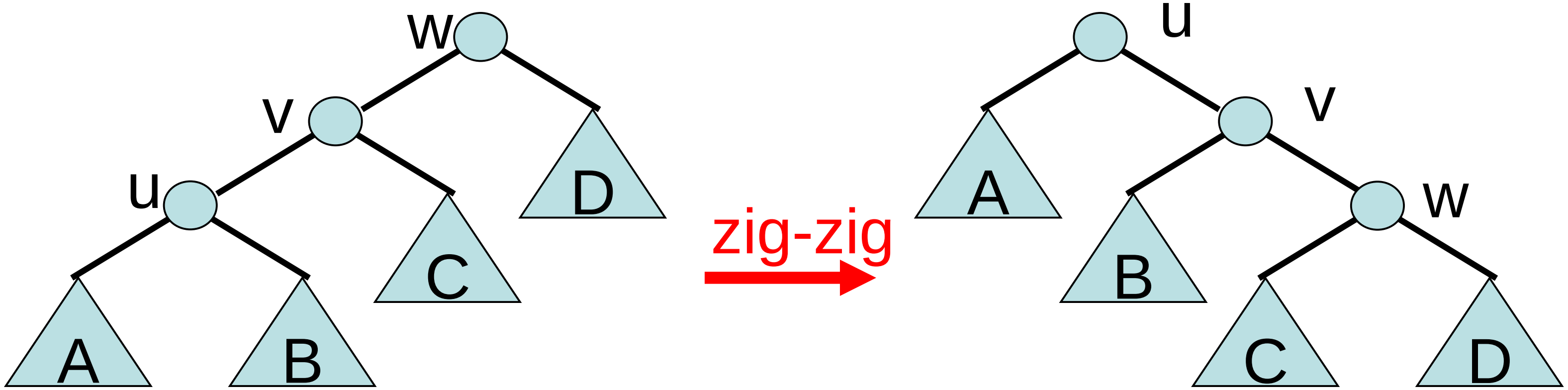
$\leq 1 + r'(u) + r'(v) - r(u) - r(v) \leq 1 + r'(u) - r(u)$ since $r'(v) \leq r(v)$

$\leq 1 + 3(r'(u) - r(u))$ since $r'(u) \geq r(u)$

Change in ϕ

Splay Operation

Case 2:



Amortized cost:

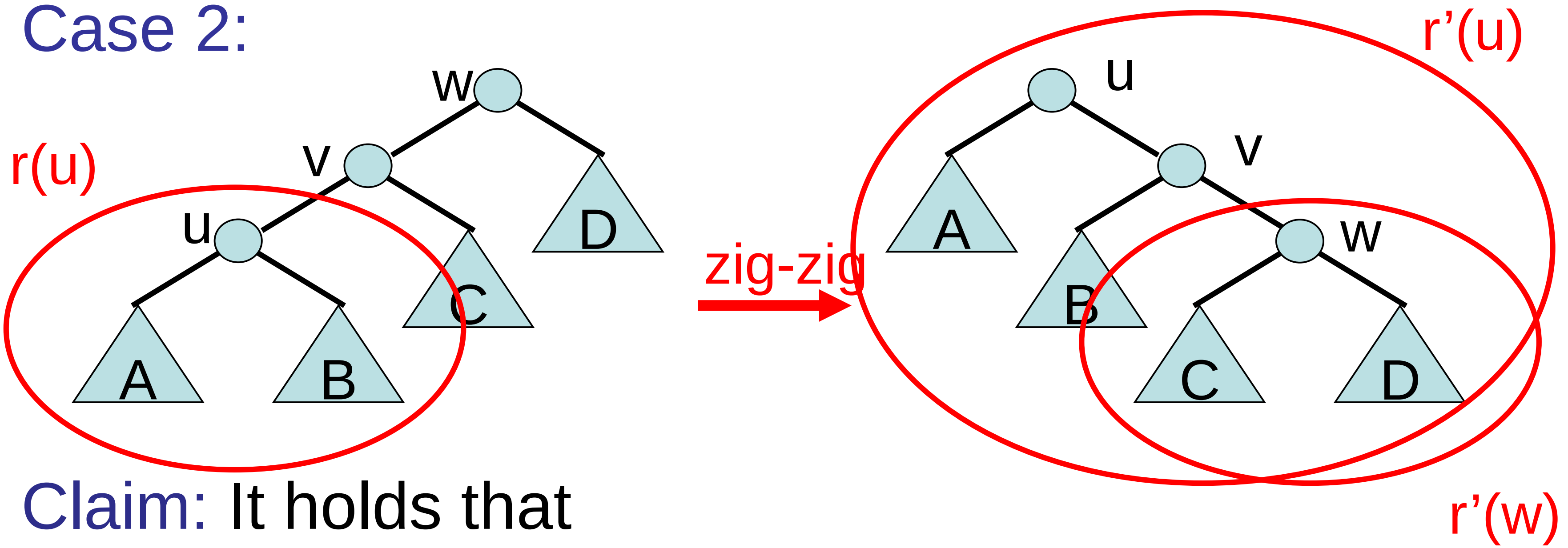
$\leq 2+r'(u)+r'(v)+r'(w)-r(u)-r(v)-r(w)$

$= 2+r'(v)+r'(w)-r(u)-r(v)$ since $r'(u)=r(w)$

$\leq 2+r'(u)+r'(w)-2r(u)$ since $r'(u)\geq r'(v)$ and $r(v)\geq r(u)$

Splay Operation

Case 2:



Claim: It holds that

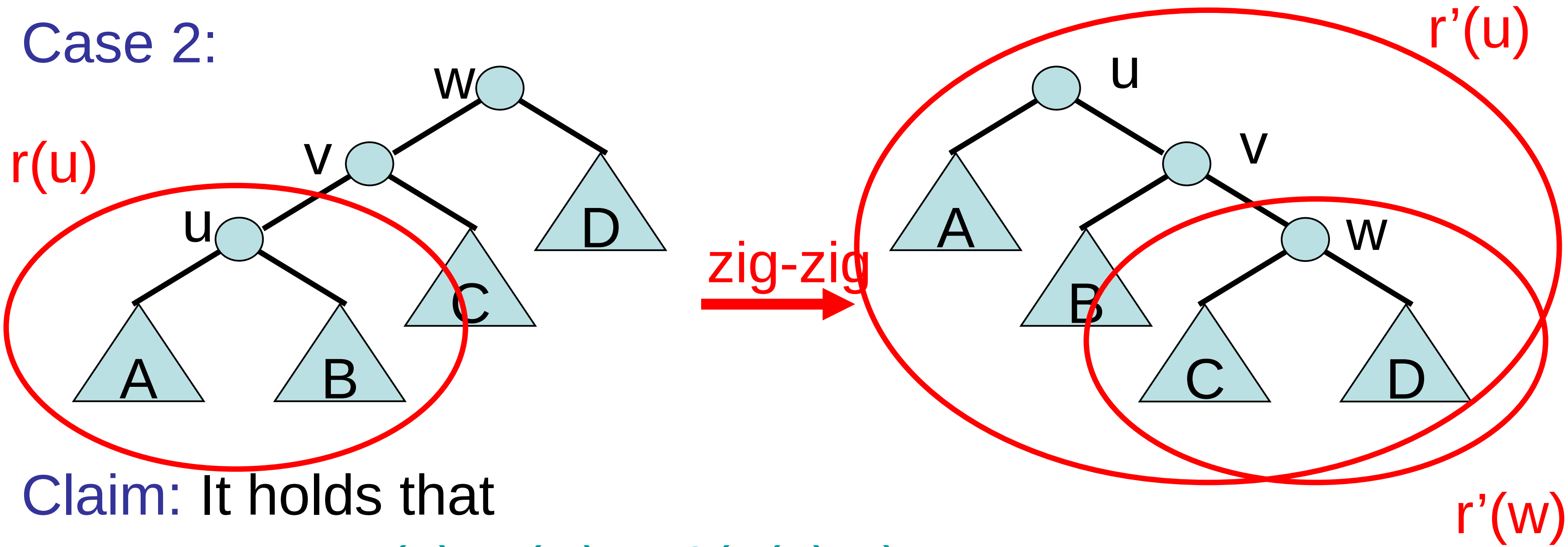
$$2+r'(u)+r'(w)-2r(u) \leq 3(r'(u)-r(u))$$

i.e.

$$r(u)+r'(w) \leq 2(r'(u)-1)$$

Splay Operation

Case 2:



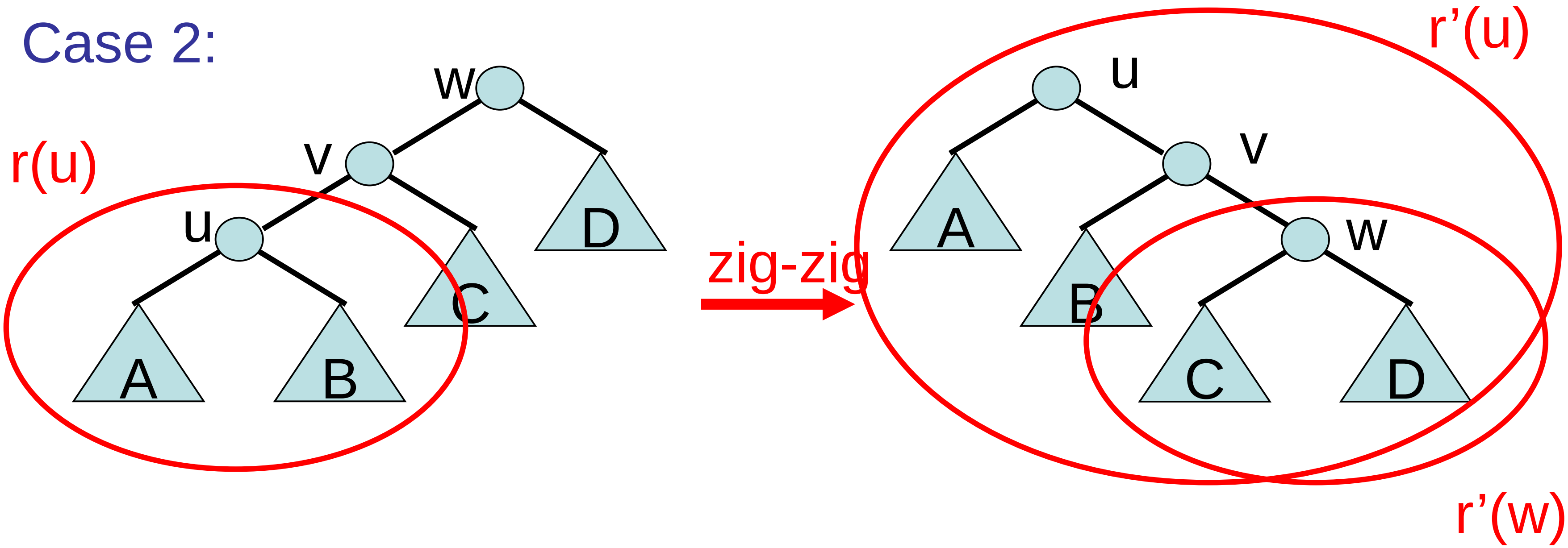
Claim: It holds that

$$r(u)+r'(w) \le 2(r'(u)-1)$$

- Recall: there are $0<x,y<1$ and a scaling factor $c>0$ with $r(u)=\log(c\cdot x)$, $r'(w)=\log(c\cdot y)$, and $r'(u)\ge\log(c(x+y))$.
- Hence, the claim holds if $\log(c\cdot x)+\log(c\cdot y) \le 2(\log(c(x+y))-1)$ for all $0<x,y<1$ and $c>0$.

Splay Operation

Case 2:



- For all $0<x,y<1$ and $c>0$ holds:

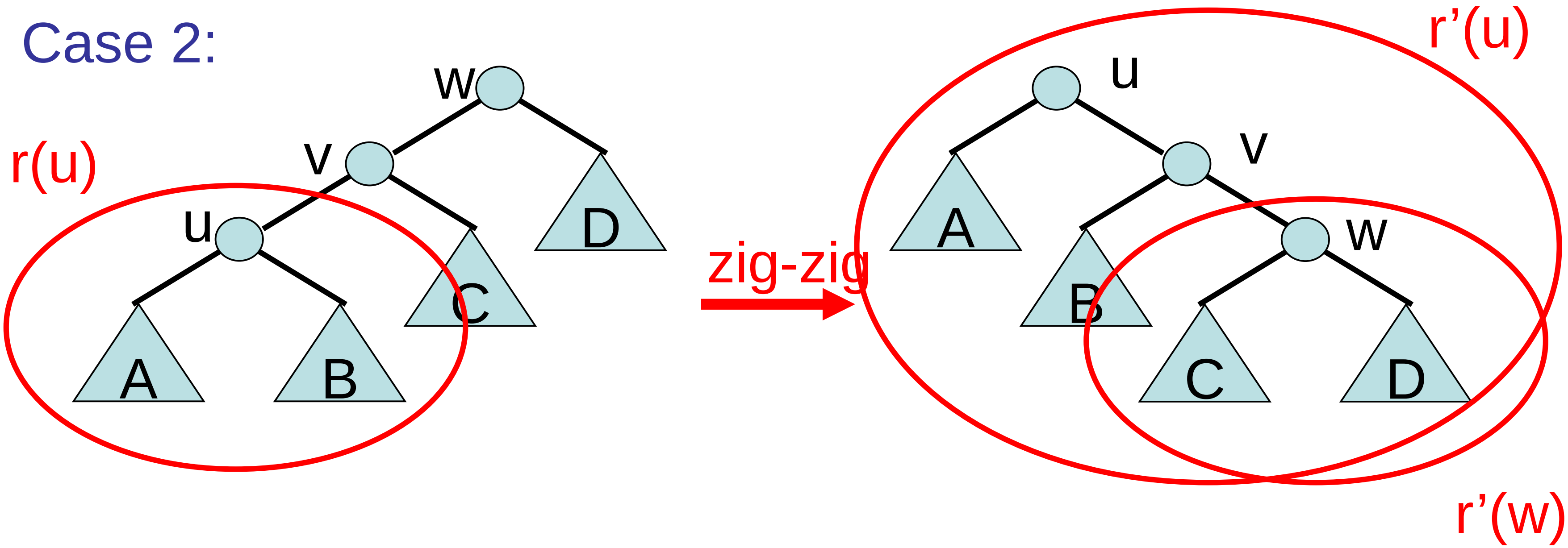
  $\log(c\cdot x)+\log(c\cdot y) \leq 2(\log(c(x+y))-1)$

  $\Leftrightarrow \log(x)+\log(y) \leq 2(\log(x+y)-1)$

- W.l.o.g. set c so that $c(x+y)=1$. Let $x'=c\cdot x$ and $y'=c\cdot y$.

Splay Operation

Case 2:



- To show: for all $0<x',y'\leq 1$, with $x'+y'=1$:

  $\log(x')+\log(y') \leq 2(\log(1)-1) = -2$

- Or more generally: show for $f(x,y)=\log(x)+\log(y)$ that

  $f(x,y)\leq -2$ for all $x,y>0$ with $x+y\leq 1$

Splay Operation

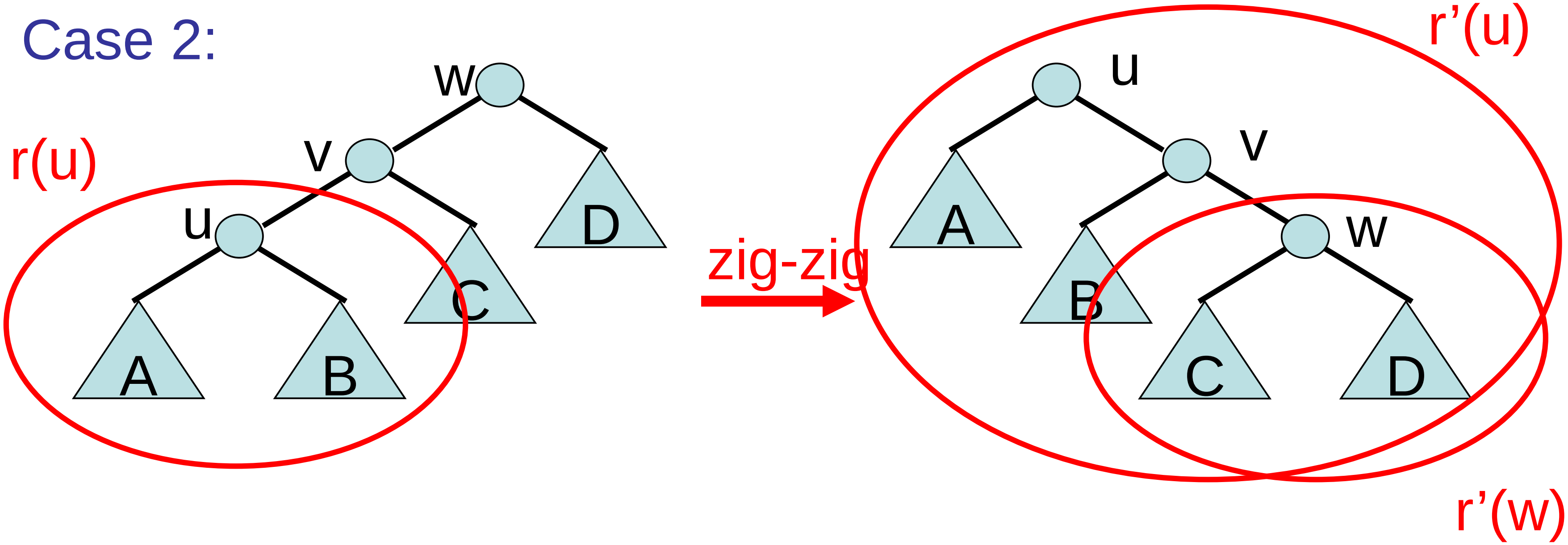
Lemma 3.2: In the area $x,y>0$ with $x+y\leq 1$, the function $f(x,y)=\log x + \log y$ has its maximum at $(½,½)$.

Proof:

- Since $\log x$ is monotonically increasing, the maximum can only lie on the line segment with $x+y=1$, $x,y>0$.
- Consider determining the maximum for $g(x) = \log x + \log (1-x)$
- The only root of $g'(x) = 1/x - 1/(1-x)$ is at $x=1/2$.
- For $g''(x)= -(1/x^2 + 1/(1-x)^2))$ it holds that $g''(1/2)<0$.
- Hence, f has its maximum at $(½,½)$.

Splay Operation

Case 2:

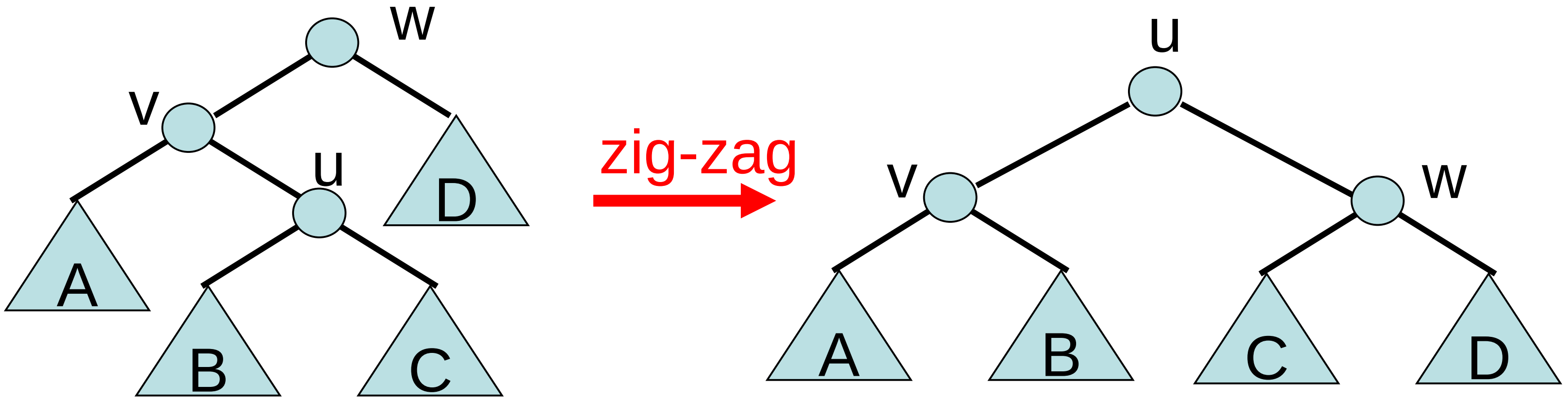


Hence, it holds that $f(x,y) \leq -2$ for all $x,y>0$ with $x+y \leq 1$, which implies the claim that

$$r(u)+r'(w) \leq 2(r'(u)-1)$$

Splay Operation

Case 3:



Amortized cost:

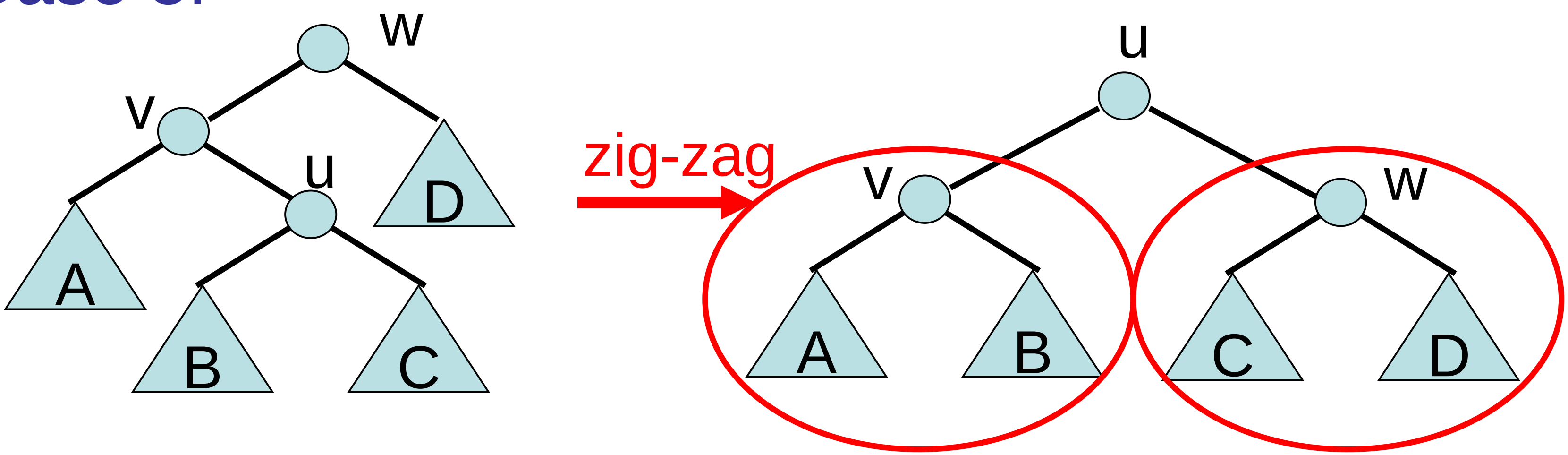
$\leq$ 2+r'(u)+r'(v)+r'(w)-r(u)-r(v)-r(w)

$\leq$ 2+r'(v)+r'(w)-2r(u) since r'(u)=r(w) and r(u)$\leq$r(v)

$\leq$ 2(r'(u)-r(u)) because…

Splay Operation

Case 3:



…it holds that:

$$2 + r'(v) + r'(w) - 2r(u) \le 2(r'(u) - r(u))$$

$$\Leftrightarrow \quad 2r'(u) - r'(v) - r'(w) \ge 2$$

$$\Leftrightarrow \quad r'(v) + r'(w) \le 2(r'(u) - 1)$$, which is true

Splay Operation

Proof of Lemma 3.1: (Follow-up)

Induction over the sequence of rotations.

- r and tw : rank and weight before the rotation
- r' und tw': rank and weight after the rotation
- For every rotation, the amortized cost is at most $1+3(r'(u)-r(u))$ (case 1) resp. $3(r'(u)-r(u))$ (cases 2 and 3)
- Summation of the costs gives at most (x: root) $1 + \sum_{Rot.} 3(r'(u)-r(u)) = 1+3(r(x)-r(u))$

Splay Operation

- Tree weight of tree T with root x:
  $tw(x) = \sum_{y \in T} w(y)$
- Rank of node x: $r(x) = \log(tw(x))$
- Potential of tree T: $\phi(T) = \sum_{x \in T} r(x)$

Lemma 3.1: Let T be a Splay tree with root x and u be a node in T. The amortized cost for splay(u,T) is at most $1+3(r(x)-r(u)) = 1+3\cdot\log(tw(x)/tw(u))$.

Corollary 3.3: Let $W=\sum_x w(x)$ and w_i be the weight of key k_i in the i-th search call. For m search operations, the amortized cost is $O(m + \sum_{i=1}^{m} \log(W/w_i))$.

Splay Tree

Theorem 3.4: The runtime for m search operations in a Splay tree T with n elements is at most

$$O(m+(m+n)\log n).$$

Proof:

- Let $w(x) = 1$ for all nodes x in T.
- Then $W=n$ and $r(x) \leq \log W = \log n$ for all x in T.
- Recall: for a sequence F of operations, the total runtime satisfies $T(F) \leq A(F) + \phi(s_0)$ for any amortized cost function A and any initial state s_0
- $\phi(s_0) = \sum_{x \in T} r_0(x) \leq n \log n$
- Hence, Corollary 3.3 implies Theorem 3.4.

Splay Tree

Suppose we have a probability distribution for the search requests.

- $p(x)$: probability of searching for key x
- $H(p) = \sum_x p(x) \cdot \log(1/p(x))$: entropy of p

Theorem 3.5: The expected runtime for m search operations in a Splay tree T with n elements is at most

$$O(m \cdot H(p) + n \cdot \log n).$$

Proof:

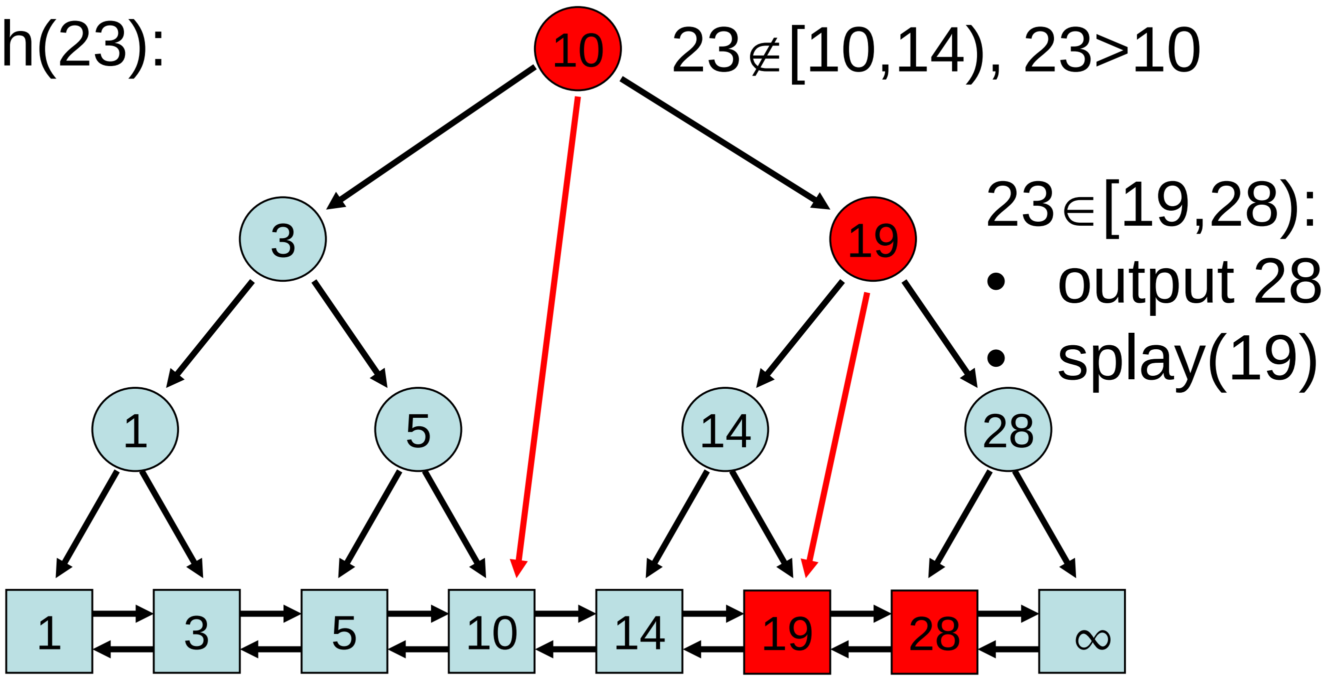
Follows from Theorem 3.4 with $w(x) = n \cdot p(x)$ for all x.

Expected runtime is $\Omega(m \cdot H(p))$ for every static binary search tree! (Optimal static tree: Huffman tree)

Splay Tree – General Case

- Instead of just exact search, the Splay tree T should also support the search for the closest successor.

search(23):



Splay Tree – General Case

- Instead of just exact search, the Splay tree T should also support the search for the closest successor.
- To obtain a low amortized time bound for that, we associate with a key x in T the search range $[x,x_+)$ (including x but excluding x_+), where x_+ is closest successor of x in T.
- Each search range $[x,x_+)$ is associated with a weight $w([x,x_+))$. Using that, we can revise Corollary 3.3 to:

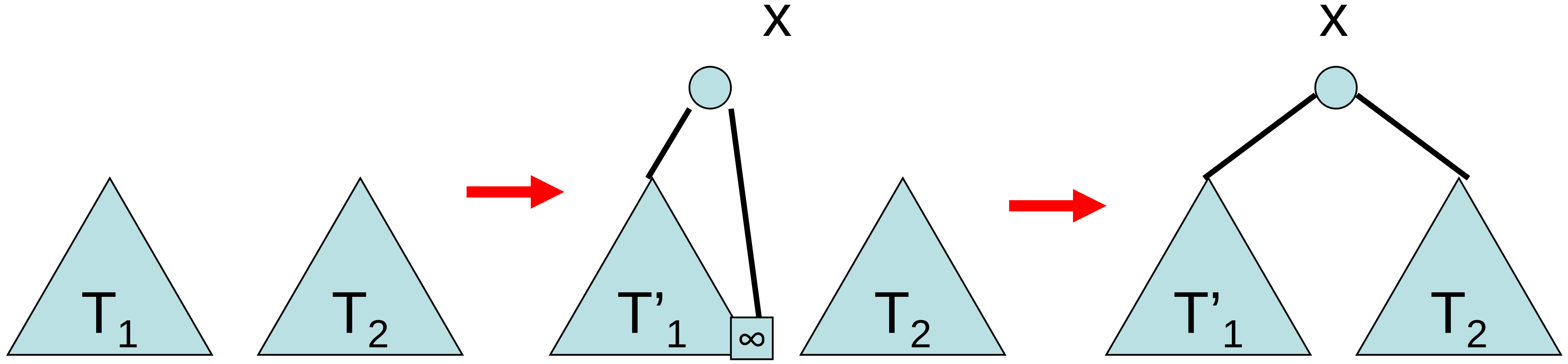
Corollary 3.3': Let $W=\sum_x w(x)$ and w_i be the weight of the range $[x,x_+)$ containing the i-th search key. For m search operations, the amortized cost is

$$O(m + \sum_{i=1}^{m} \log (W/w_i)).$$

Splay Tree Operations

Let T_1 and T_2 be two Splay trees with $key(x)<key(y)$ for all $x \in T_1$ und $y \in T_2$.

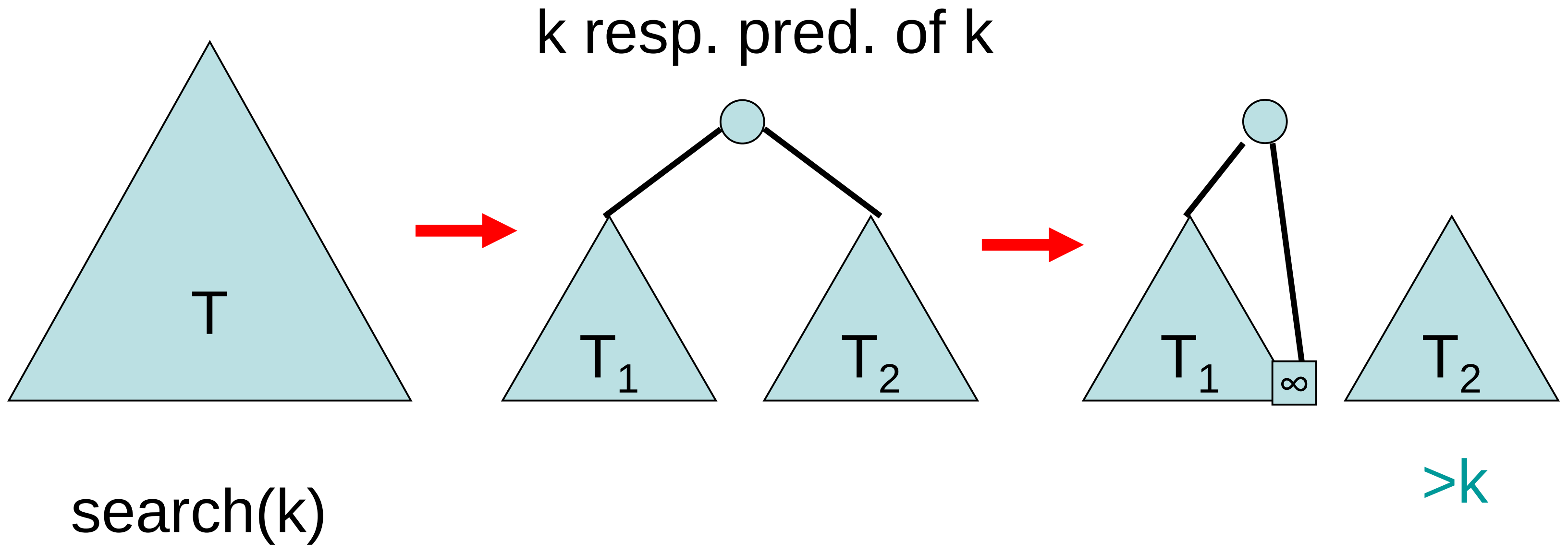
merge(T_1,T_2):



Take max. element $x<\infty$ in T_1 and splay it up to root

Splay Tree Operations

split(k,T):



Splay Tree Operations

insert(e):

- insert like in binary search tree
- Splay operation to move key(e) to the root

delete(k):

- execute search(k) (moves k to the root)
- remove root and execute merge(T_1,T_2) of the two resulting subtrees (by moving largest key of T_1 to the root)

Splay Operations

- k_-: closest predecessor $\leq k$ in T
- k_+: closest successor $>k$ in T

Theorem 3.6: The amortized cost of the following operations in the Splay tree are:

- search(k): $O(1+\log(W/w([k_-,k_+])))$
- insert(e): $O(1+\log(W/w([key(e),key(e)_+])))$
- delete(k): $O(1+\log(W/w([k,k_+])) + \log((W-w([k,k_+]))/w([k_-,k])))$

Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- hashed Patricia trie
  (loglog-search time)

Applications

(a,b)-Trees

Problem: how to maintain a balanced search tree

Idea:

- All nodes v (except for the root) have degree $d(v)$ with $a \le d(v) \le b$, where $a \ge 2$ and $b \ge 2a-1$ (otherwise this cannot be enforced)
- All leaves have the same depth

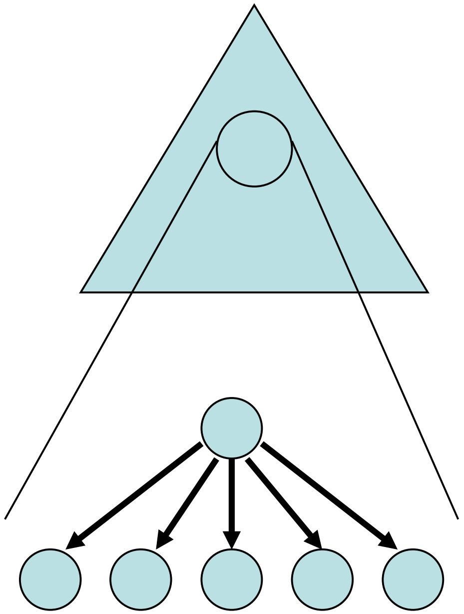
(a,b)-Trees

Formally: for a tree node v let

- $d(v)$ be the number of children of v
- $t(v)$ be the depth of v (root has depth 0)

- Form Invariant:
  For all leaves v,w: $t(v)=t(w)$

- Degree Invariant:
  For all inner nodes v
  except for root: $d(v)\in[a,b]$,
  for root r: $d(r)\in[2,b]$
  (as long as #elements >1)



(a,b)-Trees

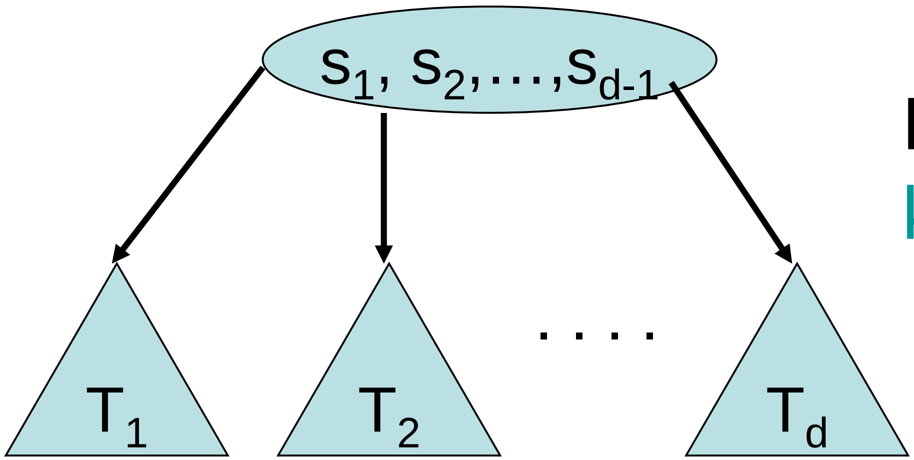
Lemma 3.10: An (a,b)-tree with n elements has depth at most $1+\lfloor \log_a (n+1)/2 \rfloor$

Proof:

- The root has degree ≥ 2 and every other inner node has degree $\geq a$.
- At depth t there are at least $2a^{t-1}$ nodes
- $n+1 \geq 2a^{t-1} \Leftrightarrow t \leq 1+\lfloor \log_a (n+1)/2 \rfloor$

(a,b)-Trees

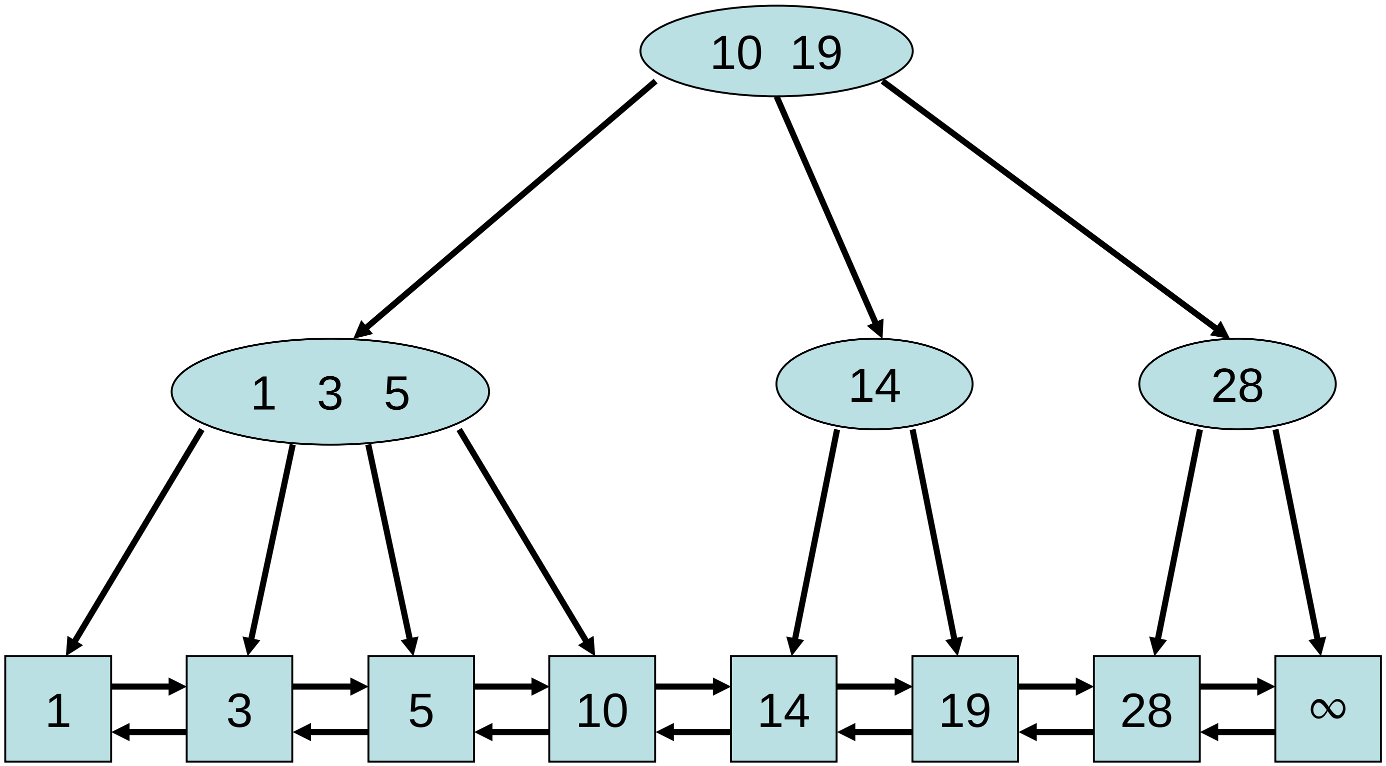
(a,b)-Tree-Rule:



For all keys k in T_i and k' in T_{i+1}: $k \leq s_i < k'$

Then search operation easy to implement.

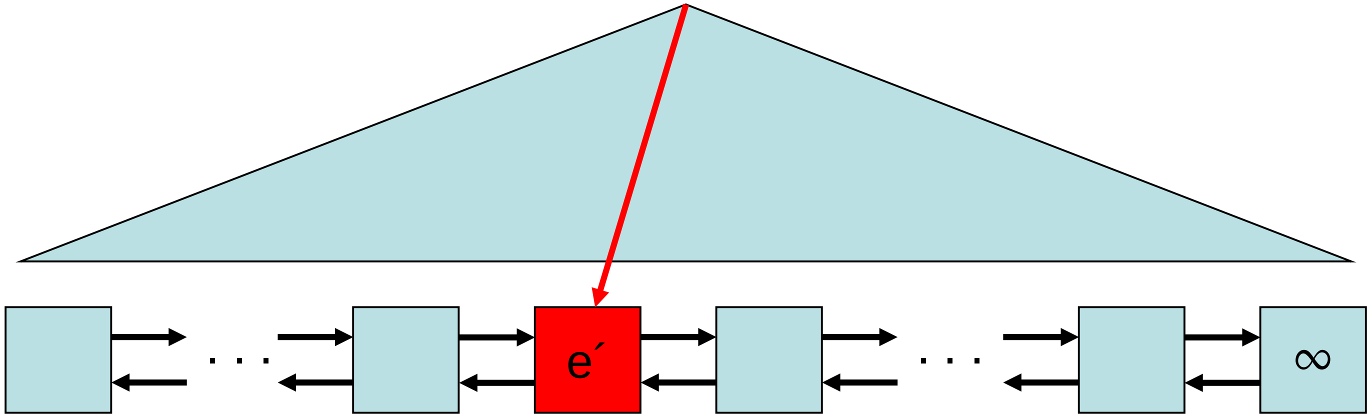
Search(9)



Insert(e) Operation

Strategy:

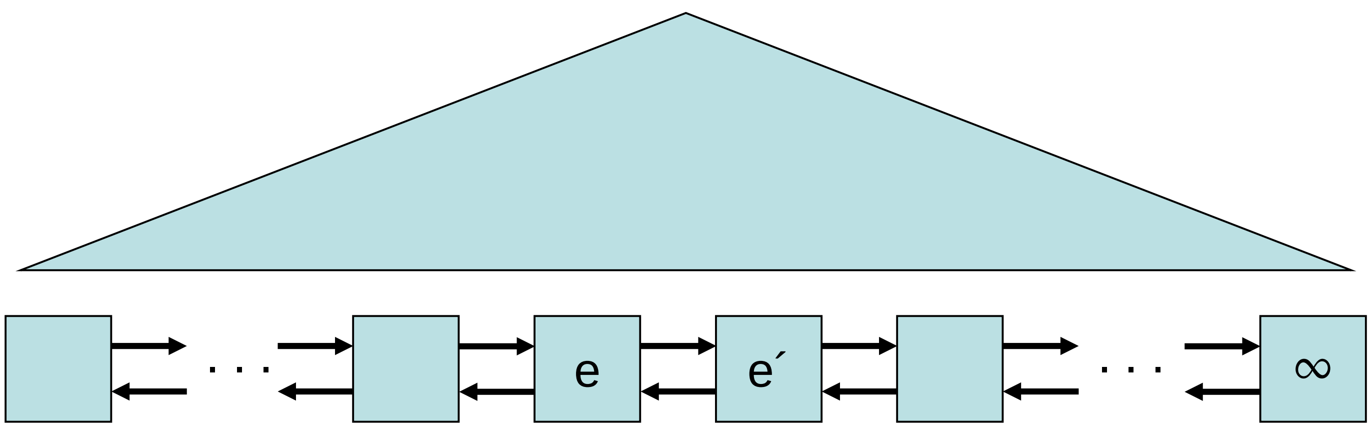
- First search(key(e)) until some e´ found in the list. If key(e´)>key(e), insert e in front of e´, otherwise replace e´ by e.



Insert(e) Operation

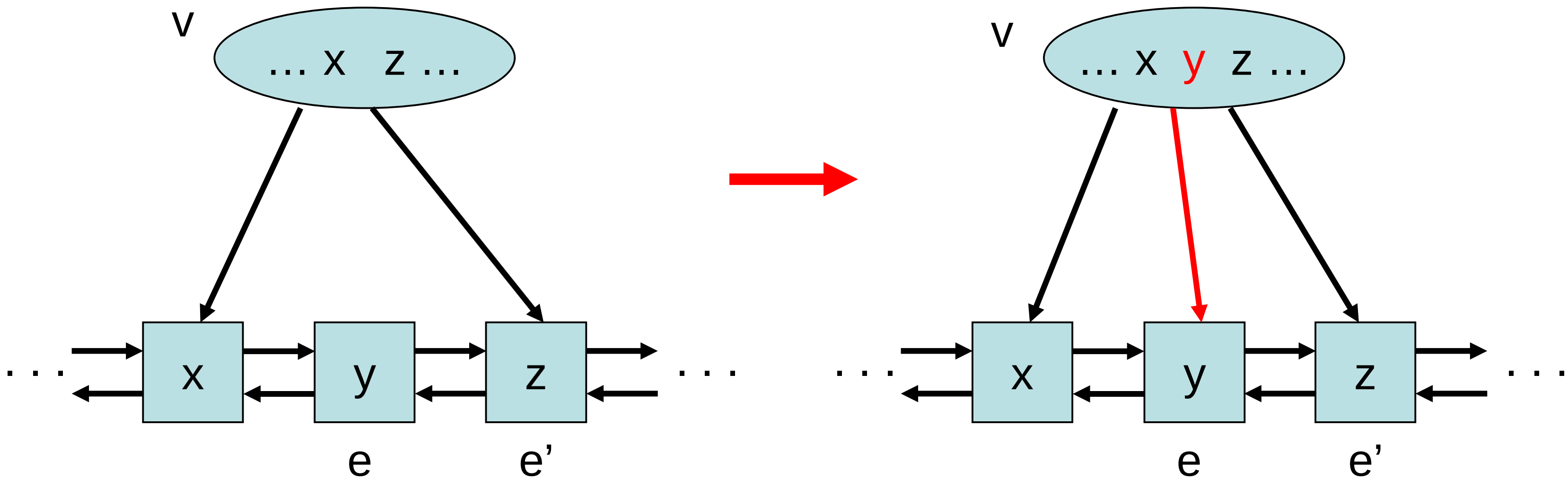
Strategy:

- First search(key(e)) until some e´ found in the list. If key(e´)>key(e), insert e in front of e´, otherwise replace e´ by e.



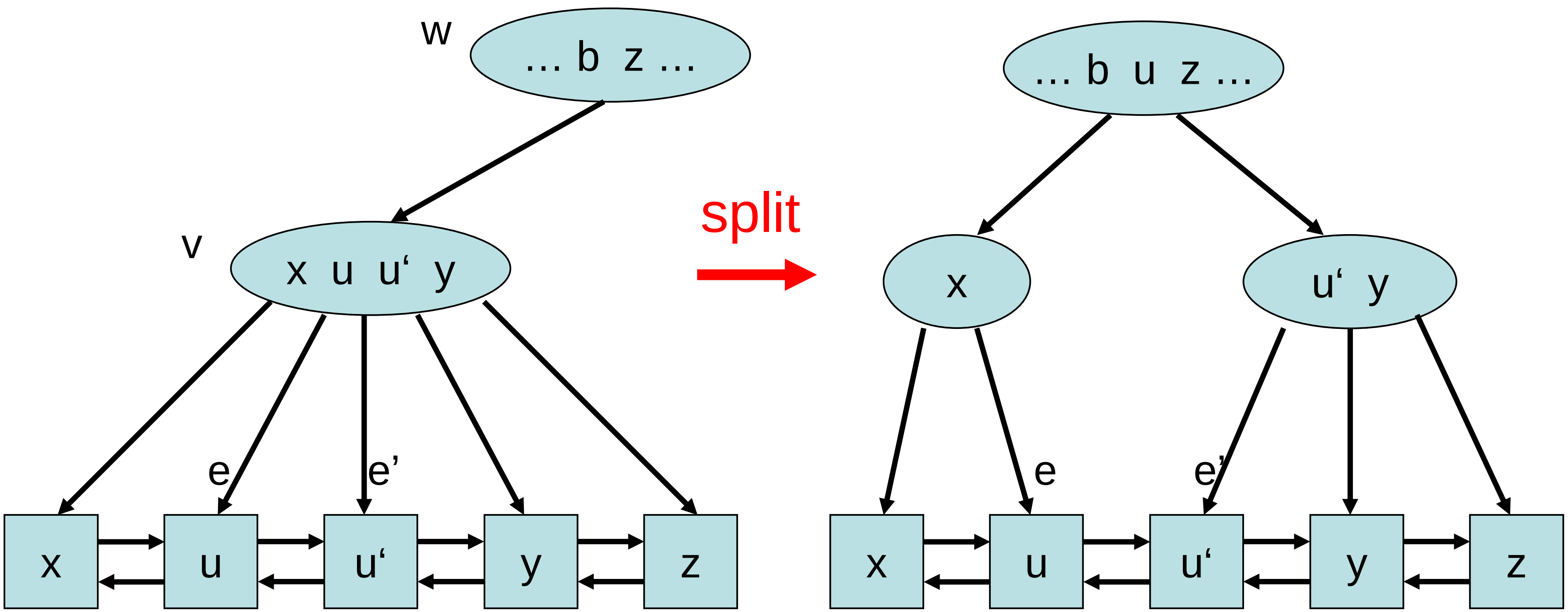
Insert(e) Operation

- Add key(e) and pointer to e in tree node v above e´. If we still have d(v)∈[a,b] afterwards, then we are done.



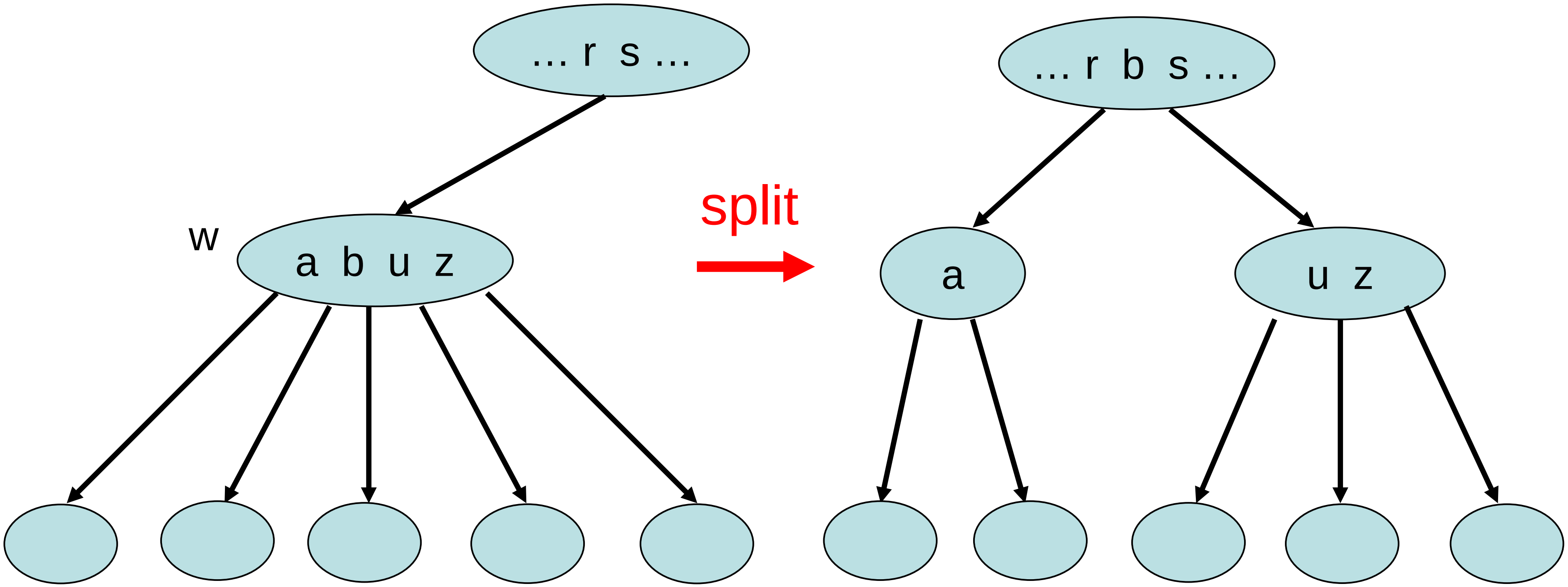
Insert(e) Operation

- If d(v)>b, then cut v into two nodes. (Example: a=2, b=4)



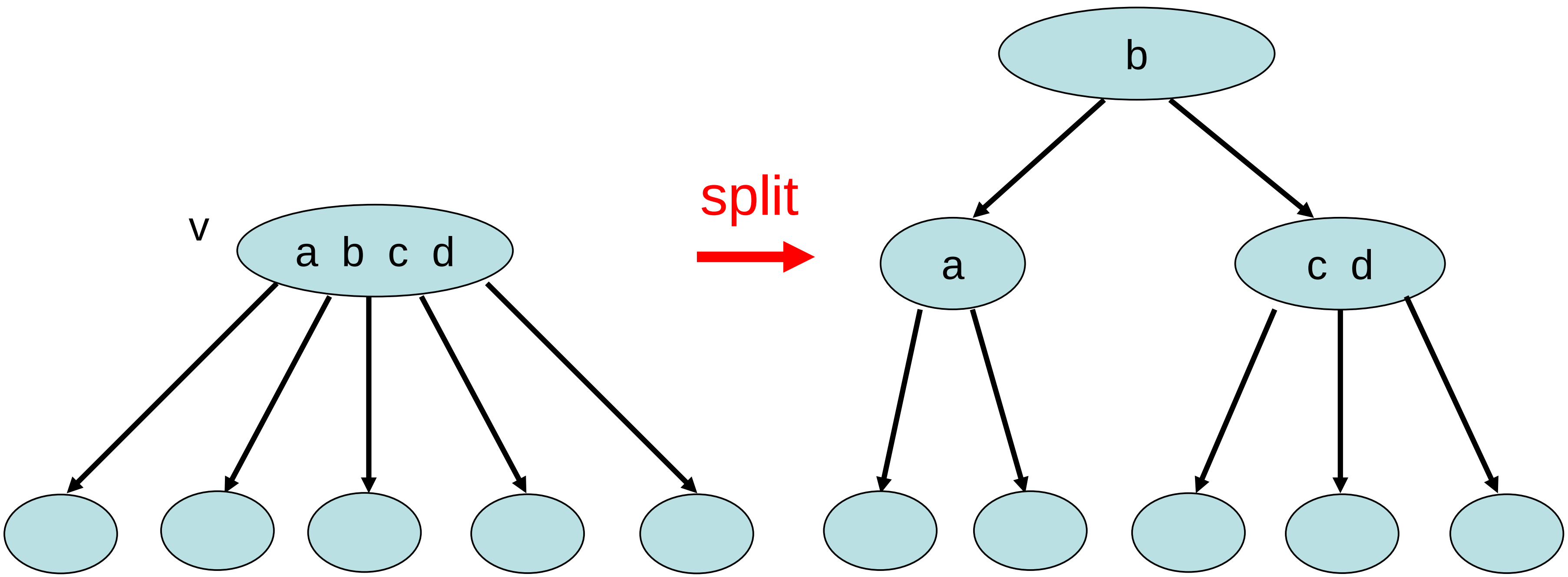
Insert(e) Operation

- If after splitting v, d(w)>b, then cut w into two nodes (and so on, until all nodes have degree ≤b or we reached the root)



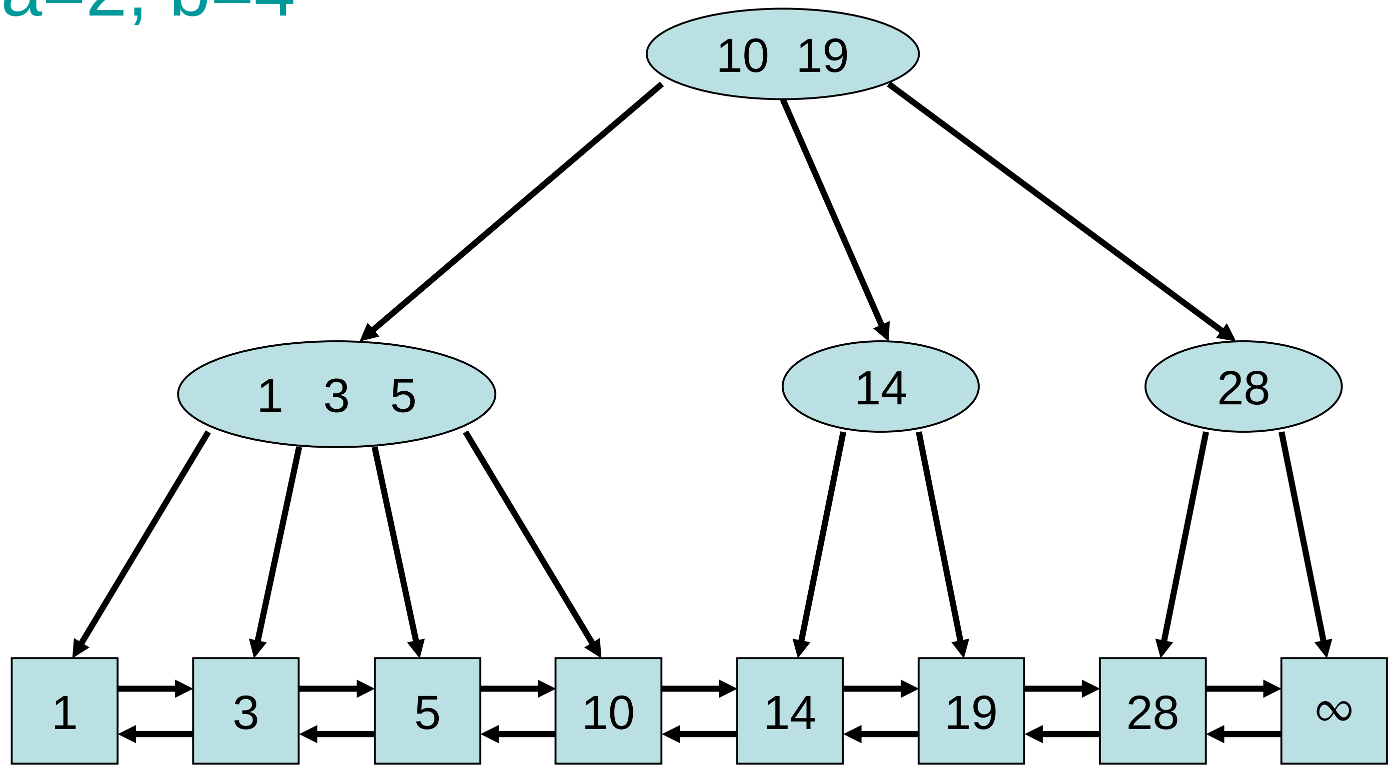
Insert(e) Operation

- If for the root v of T, d(v)>b, then cut v into two nodes and create a new root node.



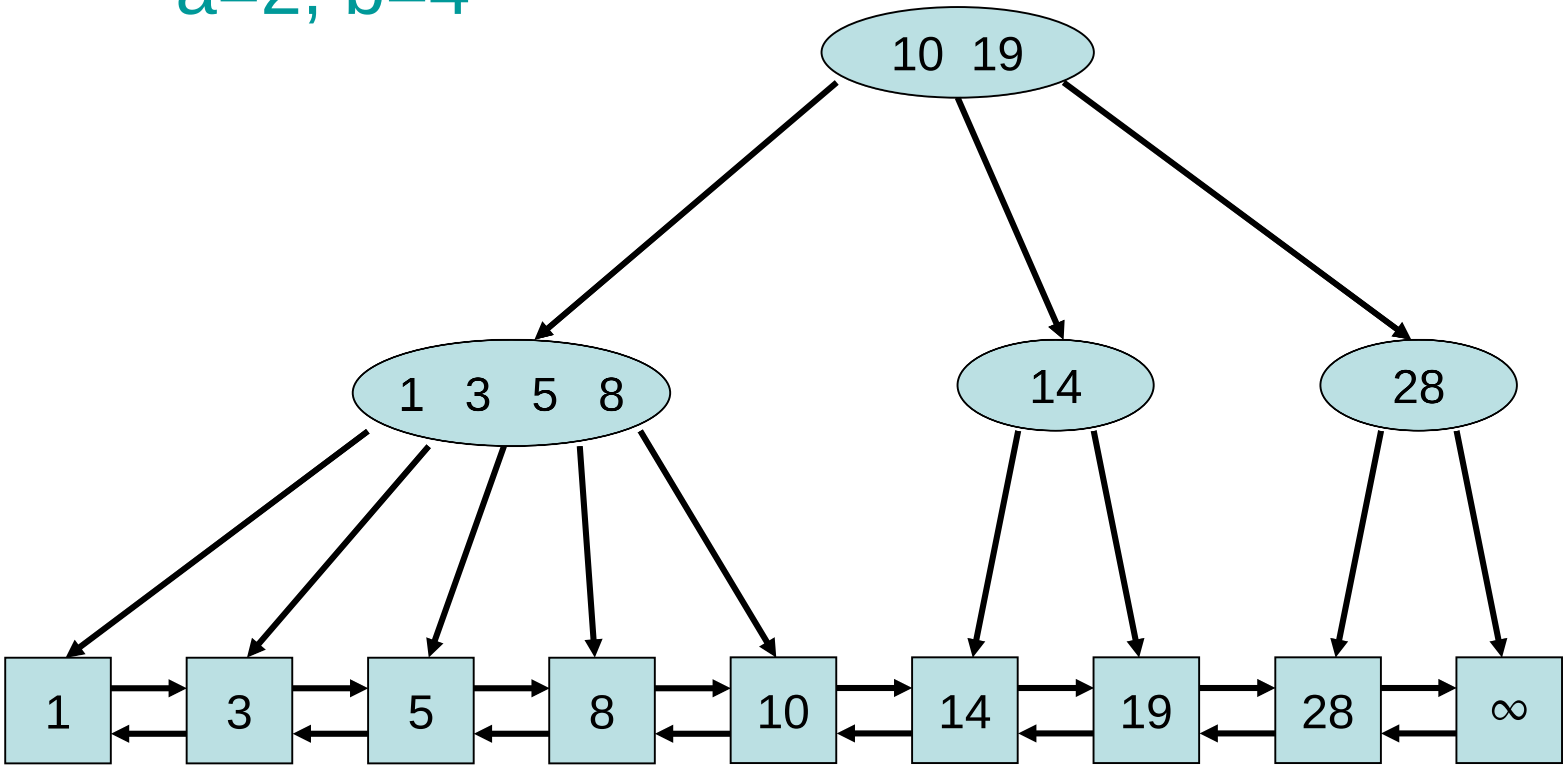
Insert(8)

a=2, b=4



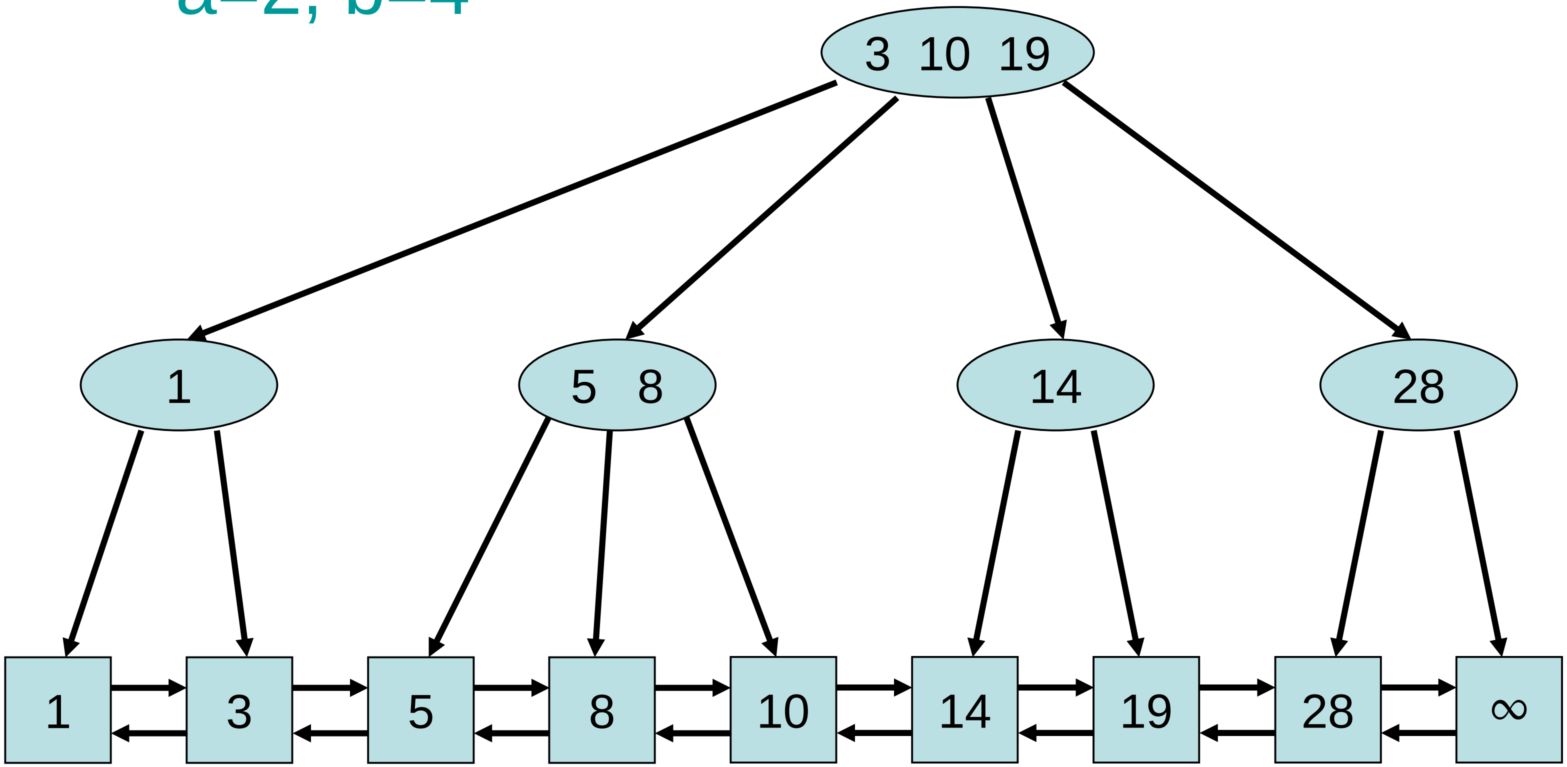
Insert(8)

a=2, b=4



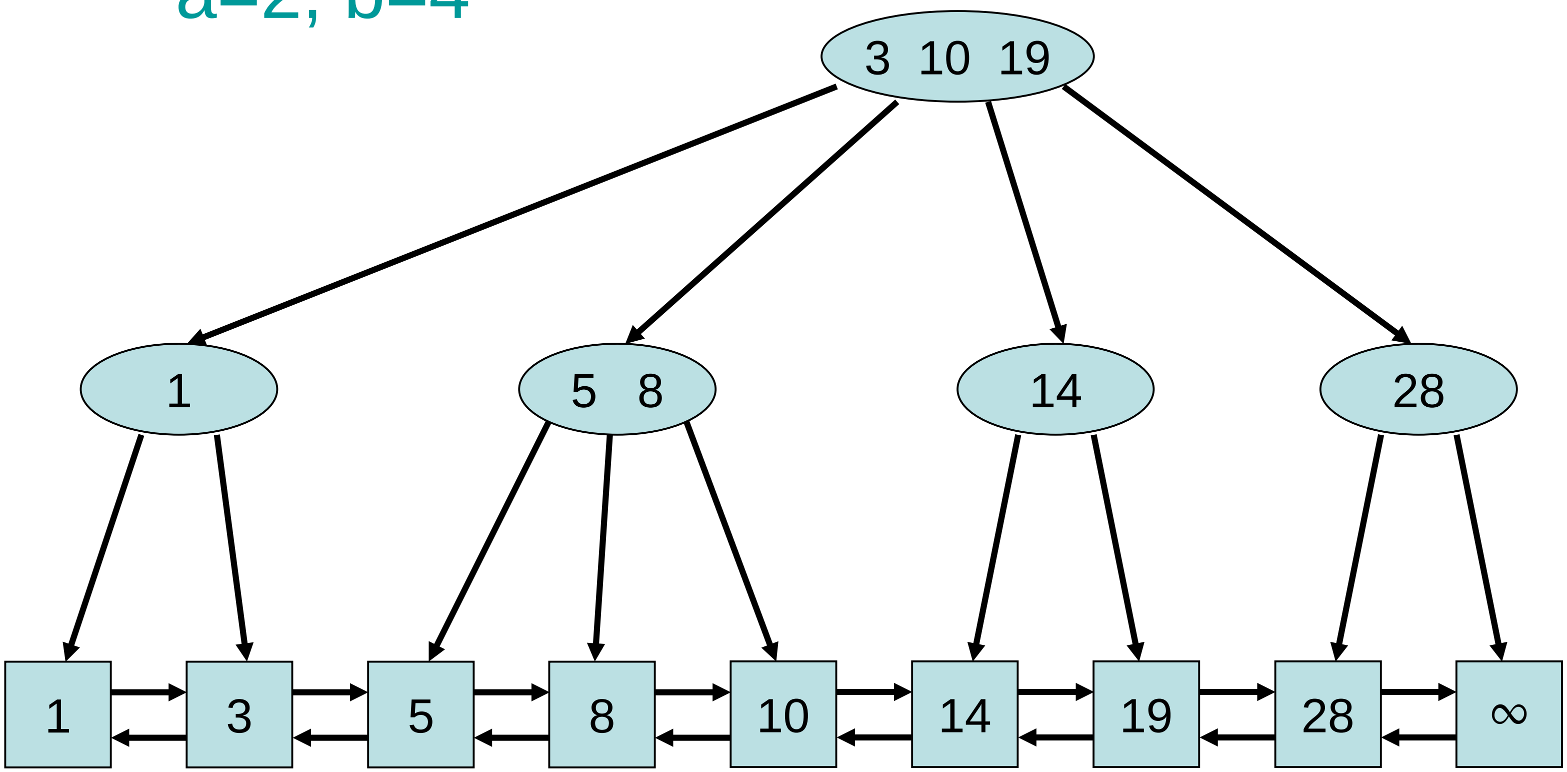
Insert(8)

a=2, b=4



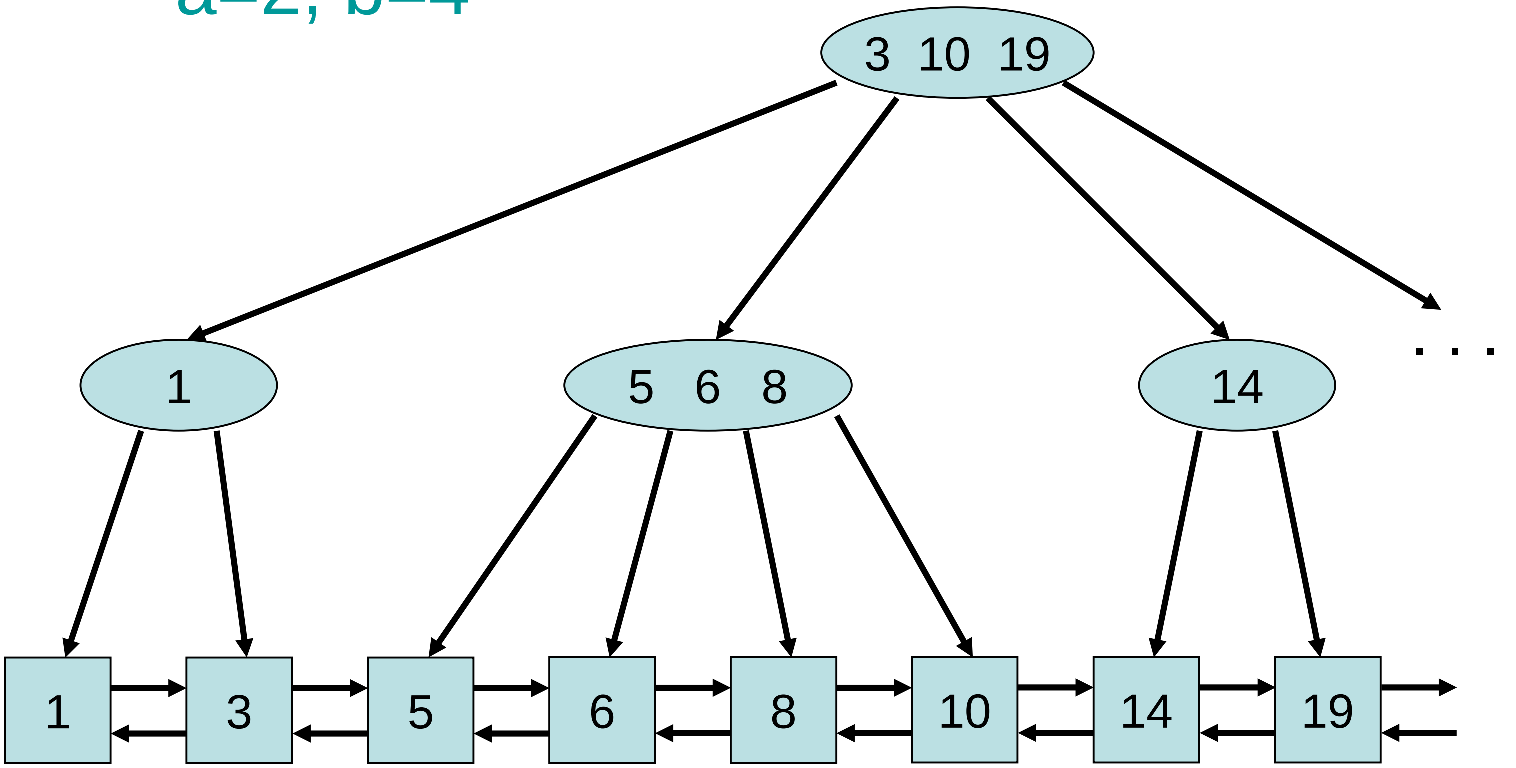
Insert(6)

a=2, b=4



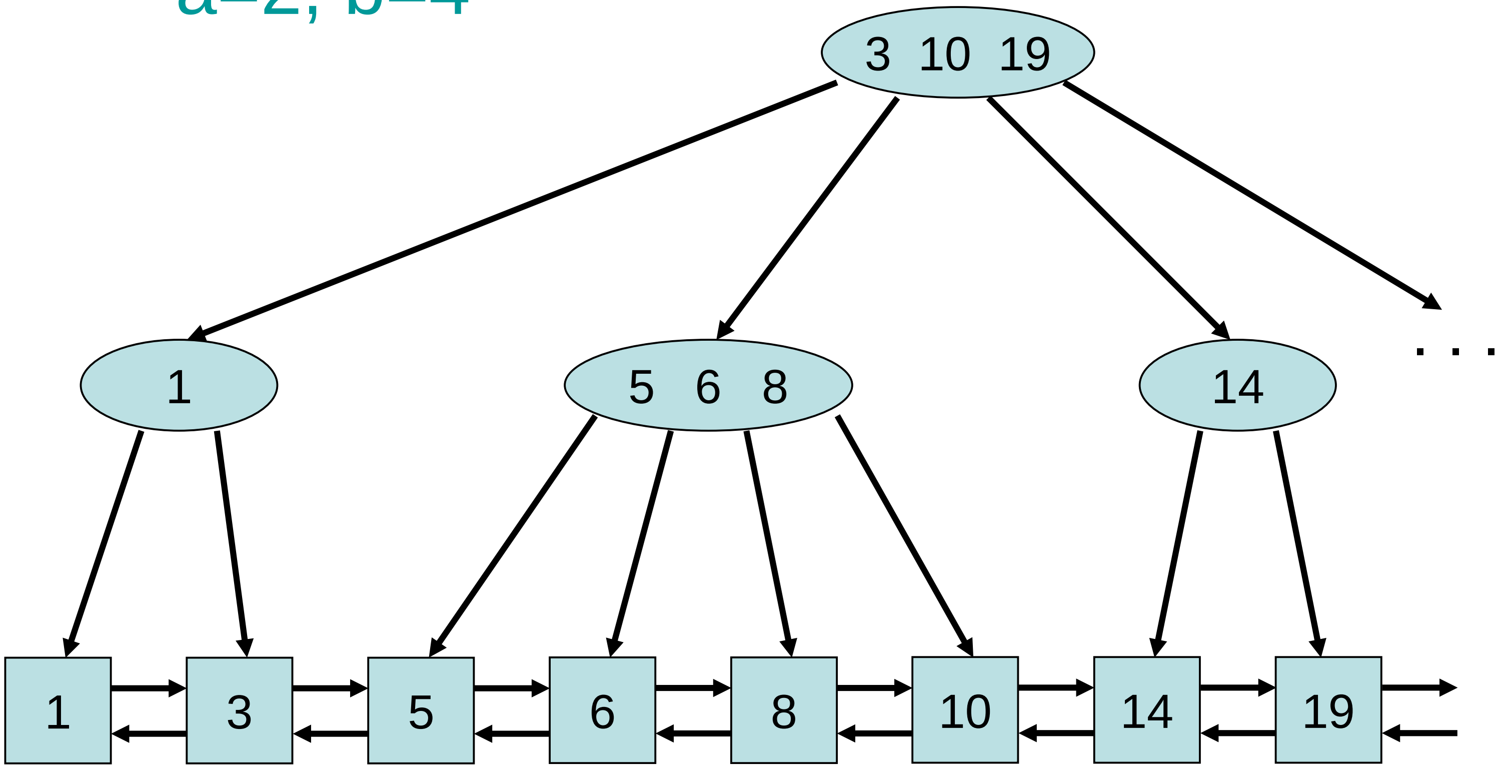
Insert(6)

a=2, b=4



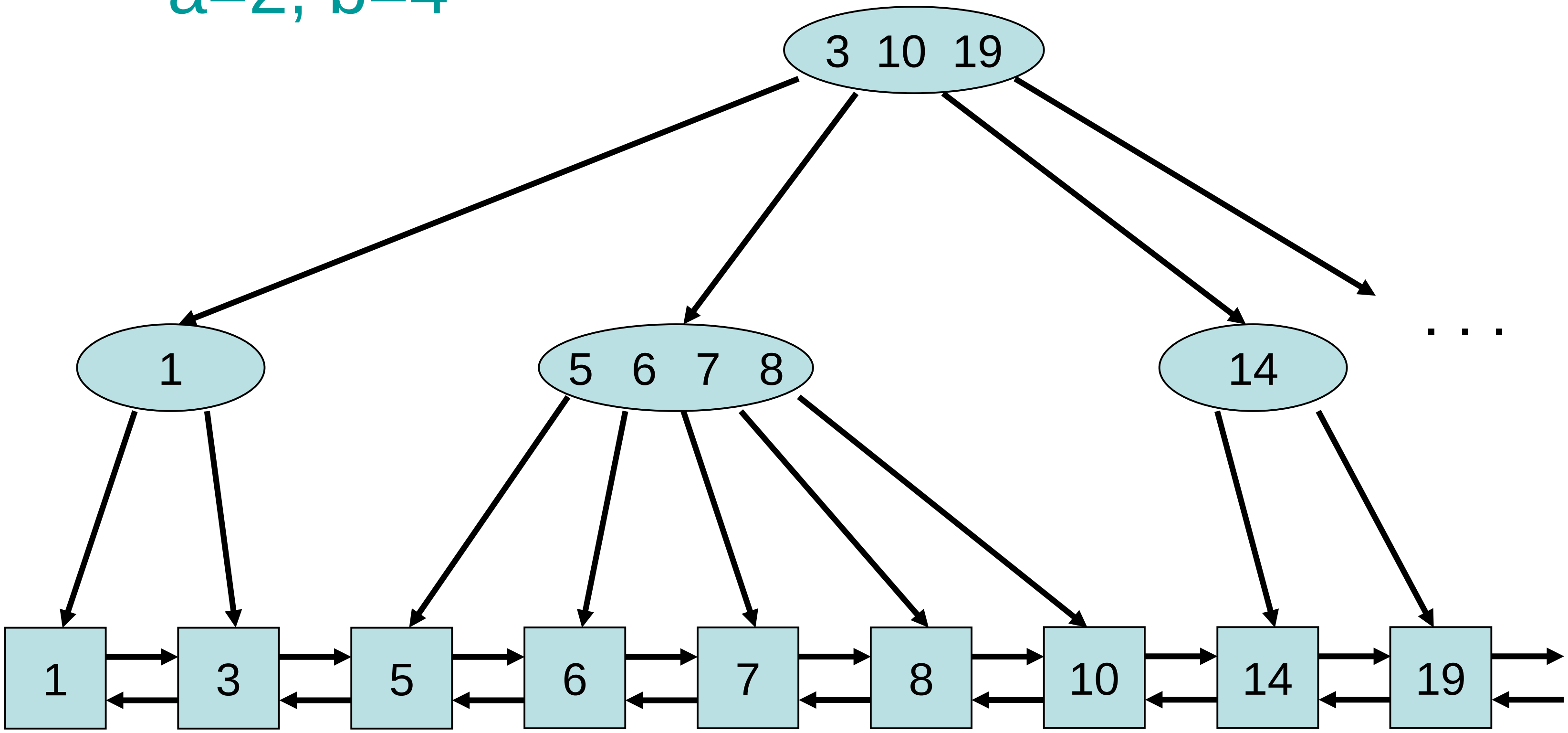
Insert(7)

a=2, b=4



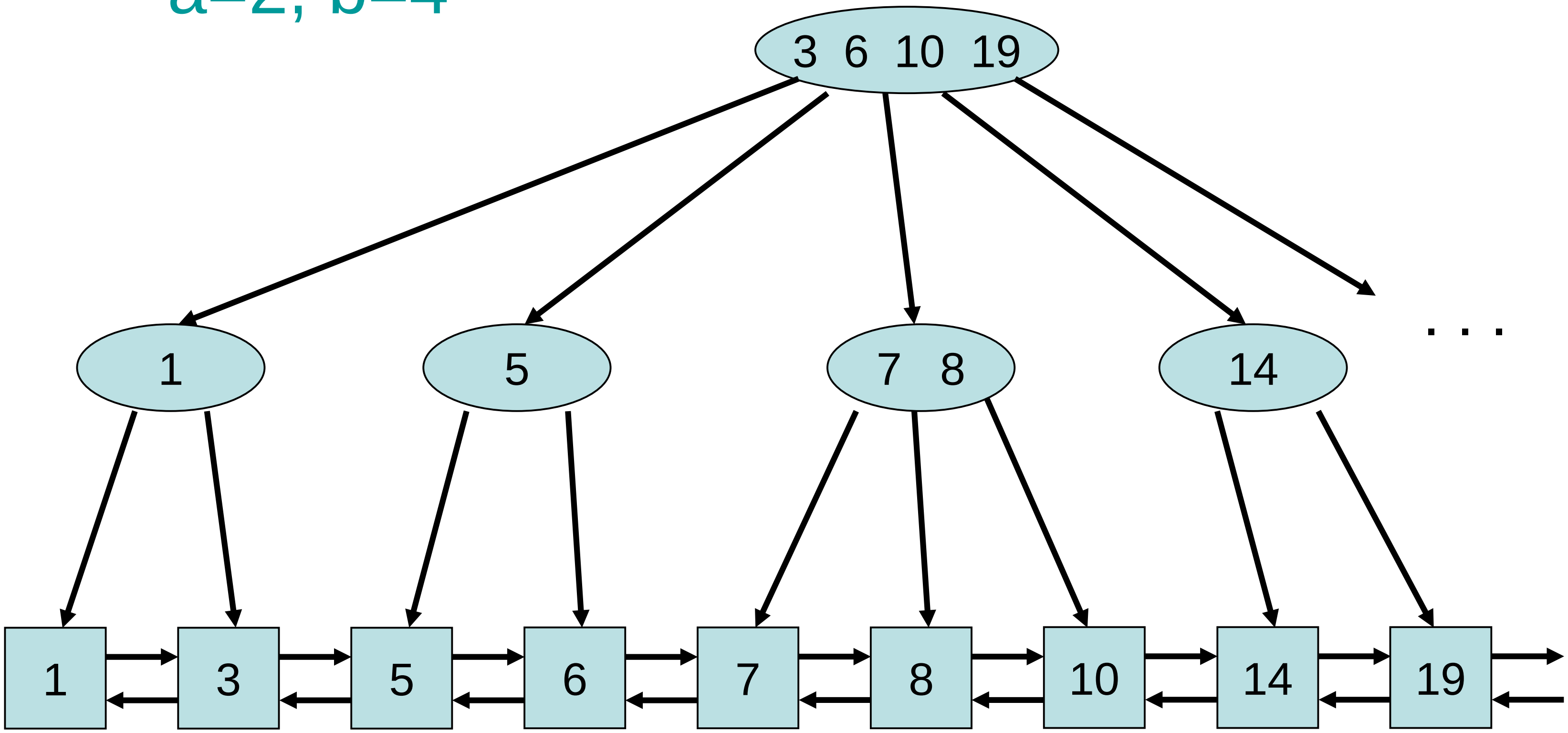
Insert(7)

a=2, b=4



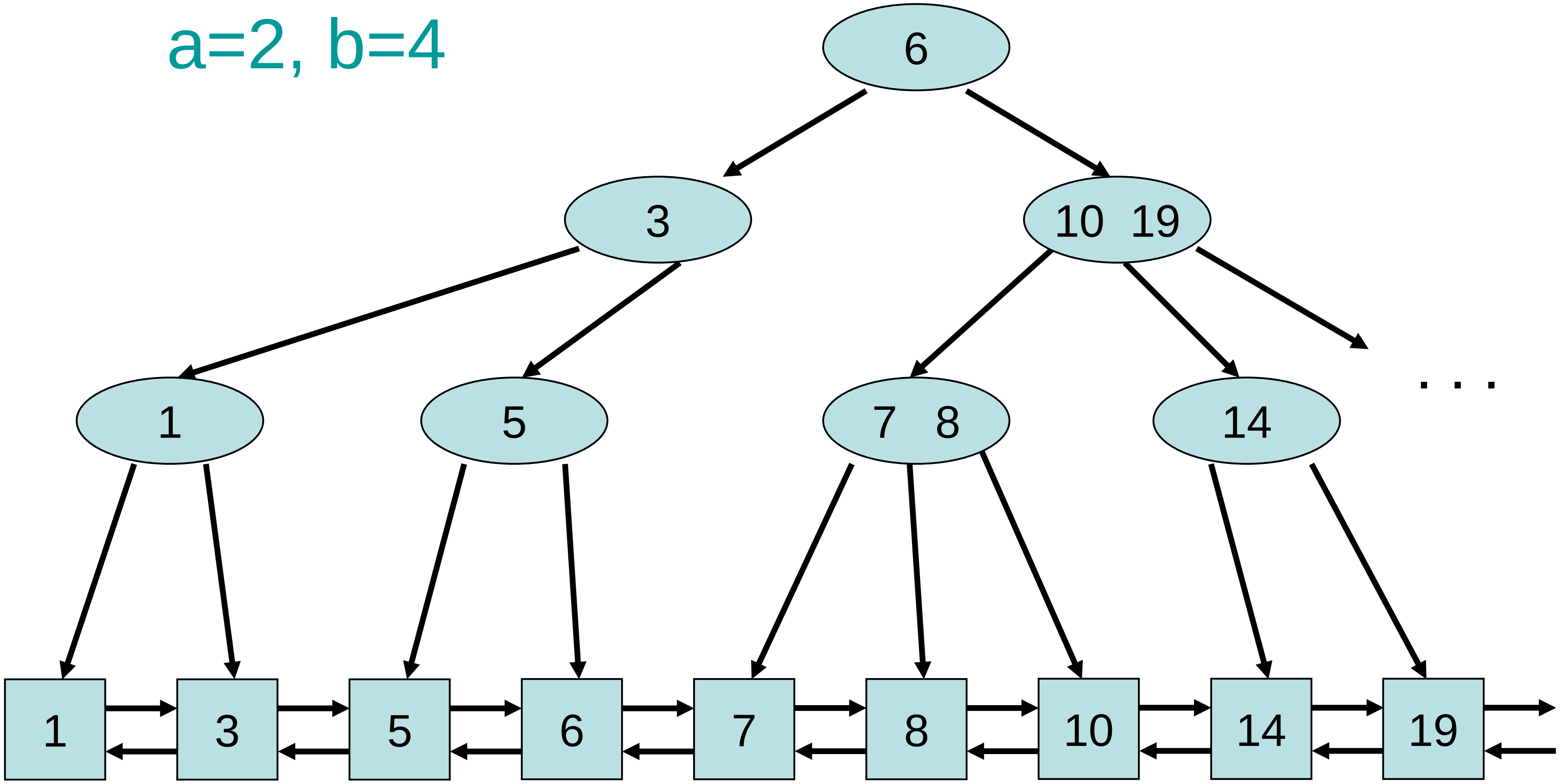
Insert(7)

a=2, b=4



Insert(7)

a=2, b=4



Insert Operation

- Form Invariant:
  For all leaves v,w: $t(v)=t(w)$
  Satisfied by Insert!

- Degree Invariant:
  For all inner nodes v except for the root:
  $d(v) \in [a,b]$, for root r: $d(r) \in [2,b]$

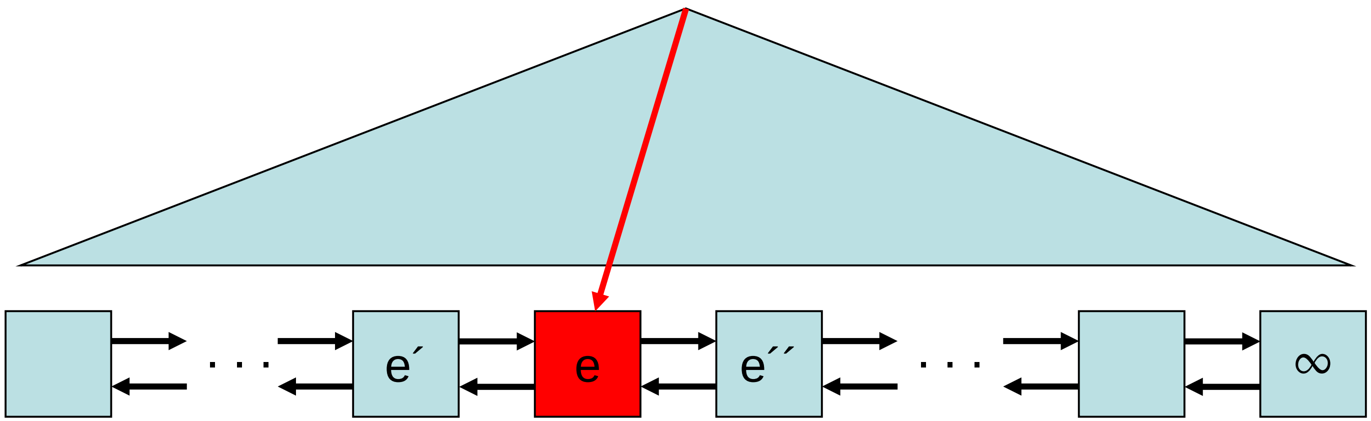
1) Insert splits nodes of degree $b+1$ into nodes of degree $\lfloor (b+1)/2 \rfloor$ and $\lceil (b+1)/2 \rceil$. If $b \geq 2a-1$, then both values are at least a.

2) If root has reached degree $b+1$, then a new root of degree 2 is created.

Delete(k) Operation

Strategy:

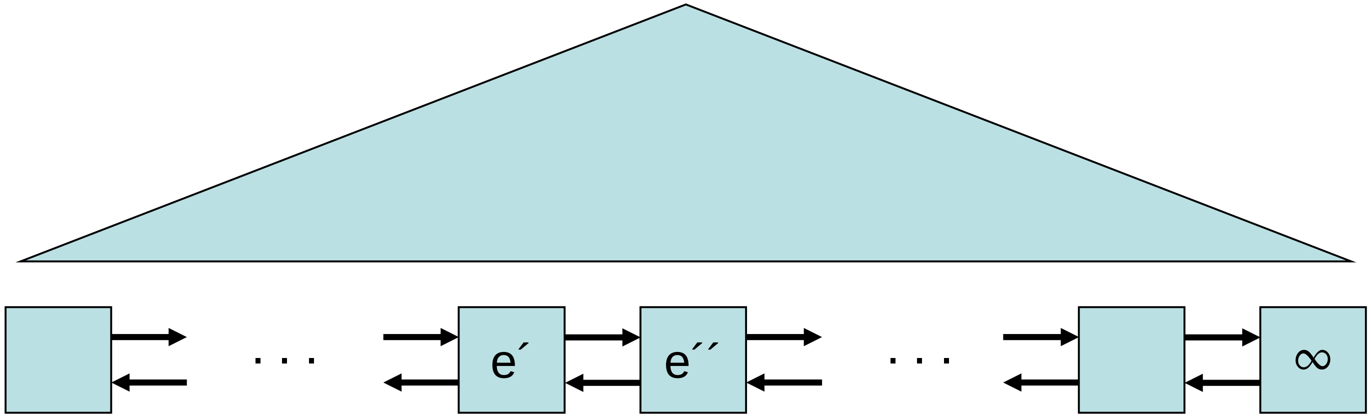
- First search(k) until some element e is reached in the list. If key(e)=k, remove e from the list, otherwise we are done.



Delete(k) Operation

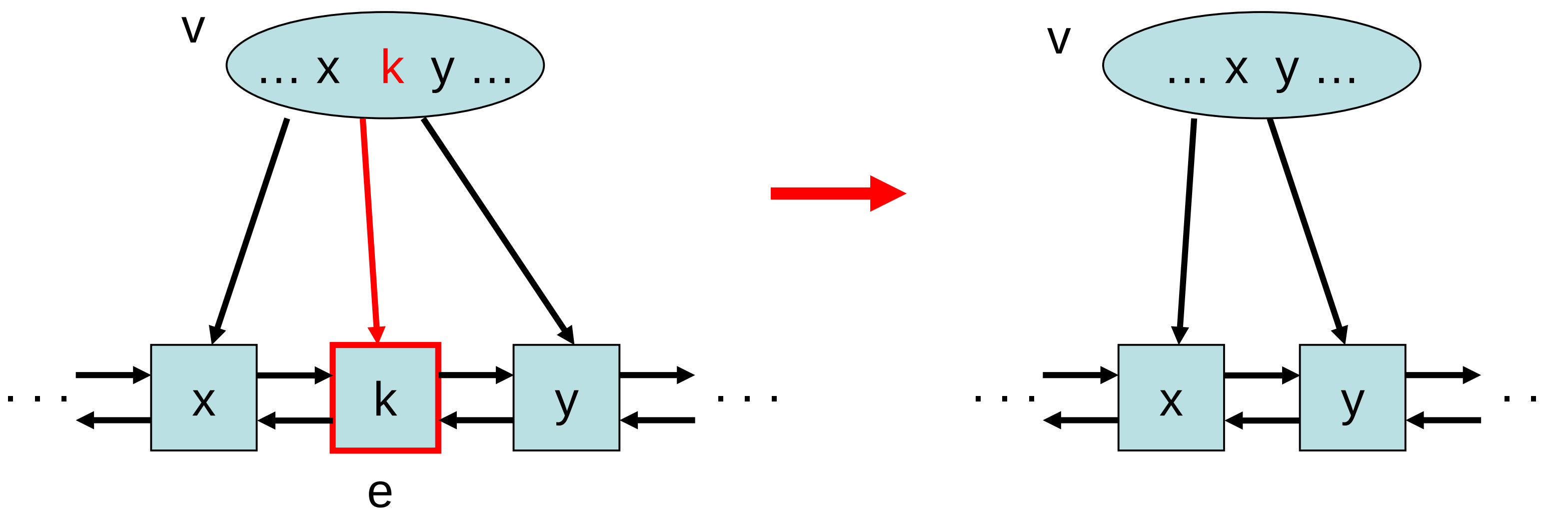
Strategy:

- First search(k) until some element e is reached in the list. If key(e)=k, remove e from the list, otherwise we are done.



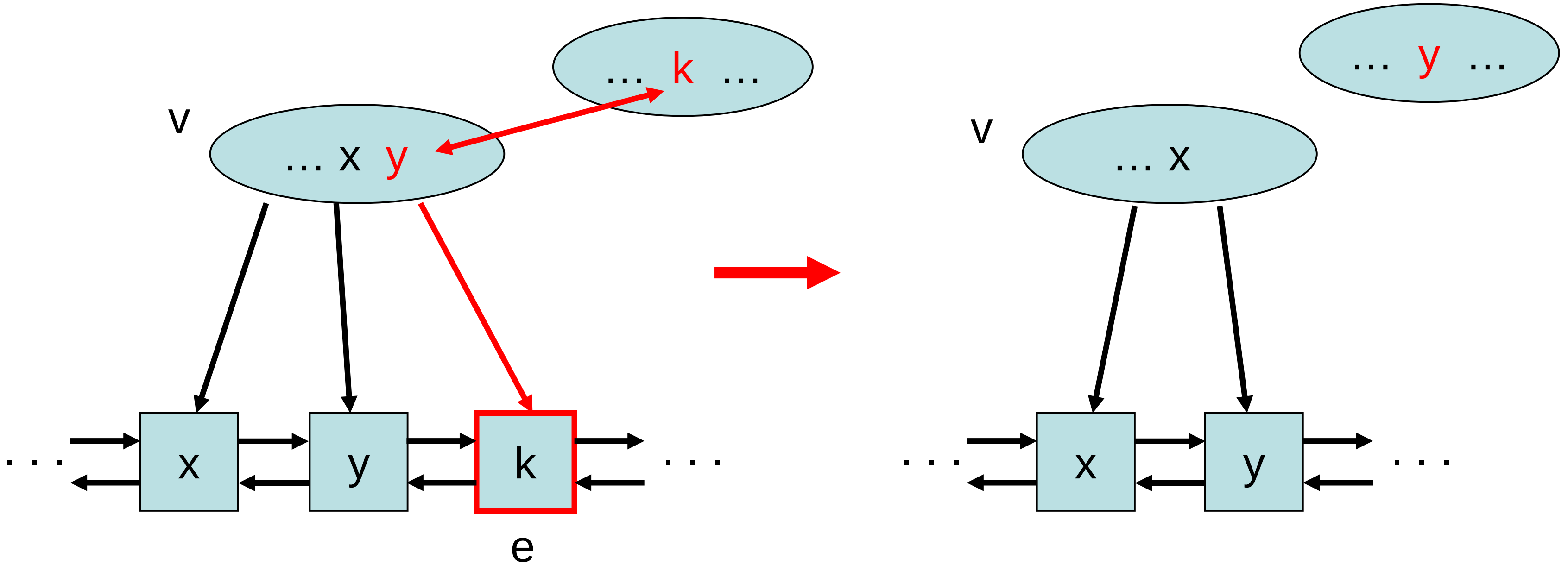
Delete(k) Operation

- Remove pointer to e and key k from the leaf node v above e. (e rightmost child: perform key exchange like in binary tree!) If afterwards we still have $d(v) \geq a$, we are done.



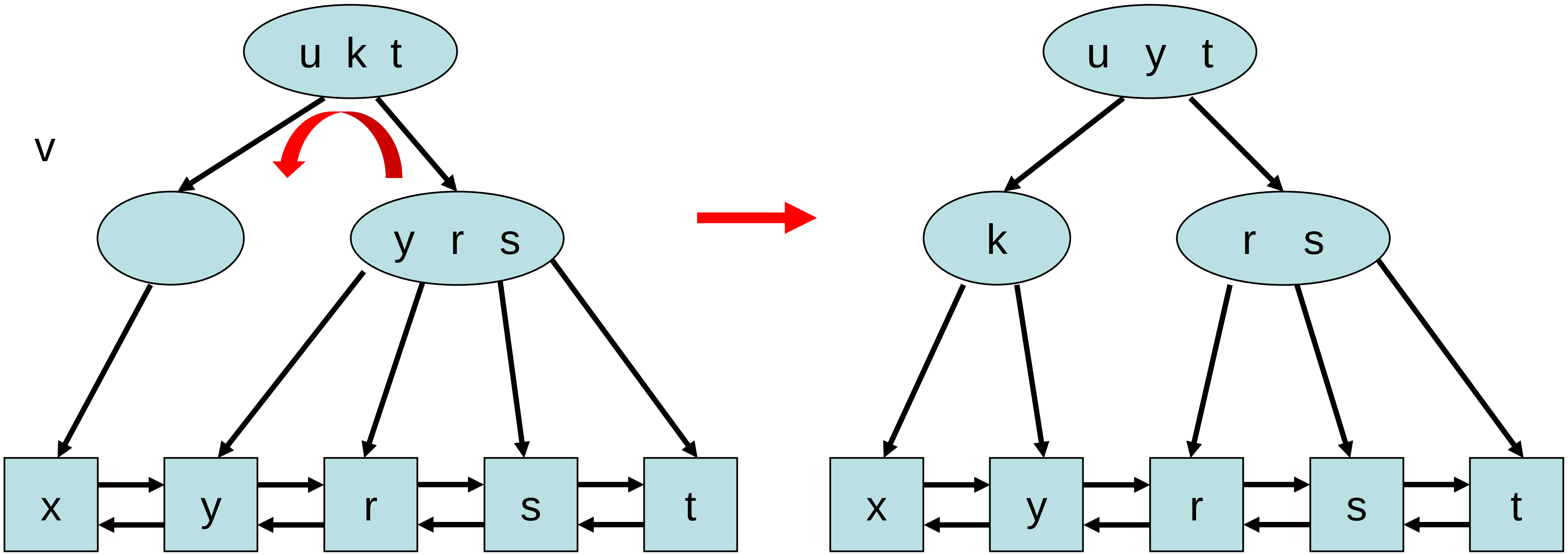
Delete(k) Operation

- Remove pointer to e and key k from the leaf node v above e. (e rightmost child: perform key exchange like in binary tree!) If afterwards we still have d(v)≥a, we are done.



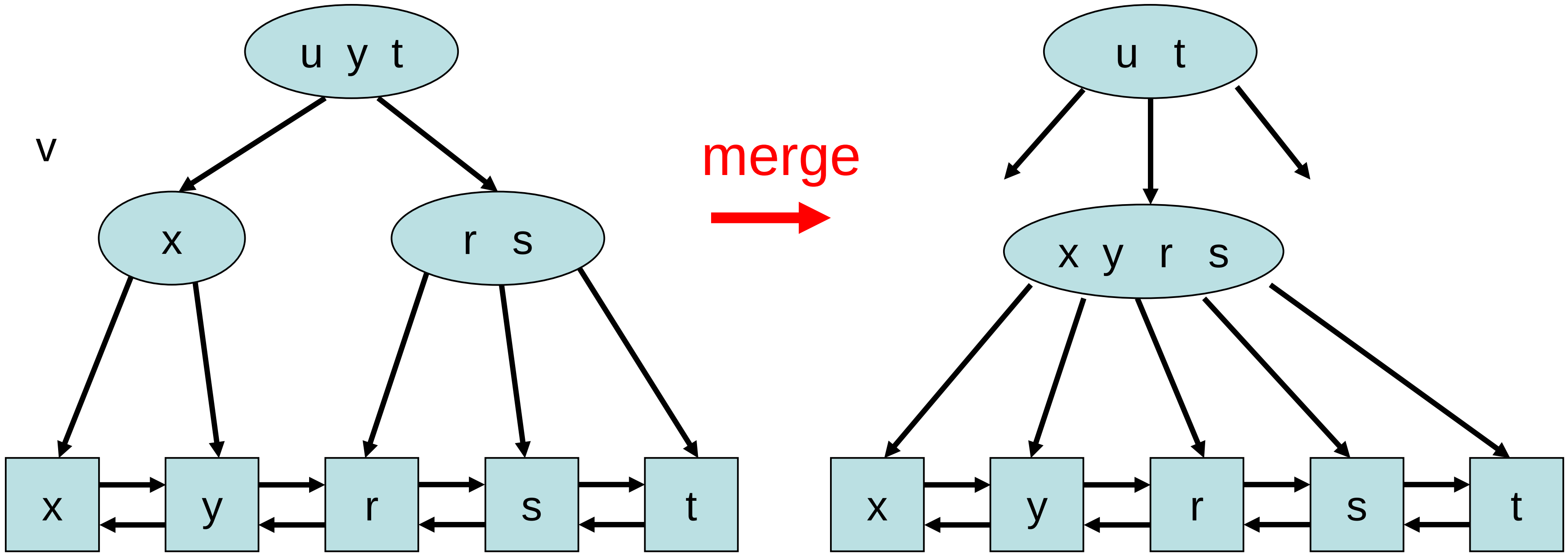
Delete(k) Operation

- If d(v)<a and the preceding or succeeding sibling of v has degree >a, steal an edge from that sibling. (Example: a=2, b=4)



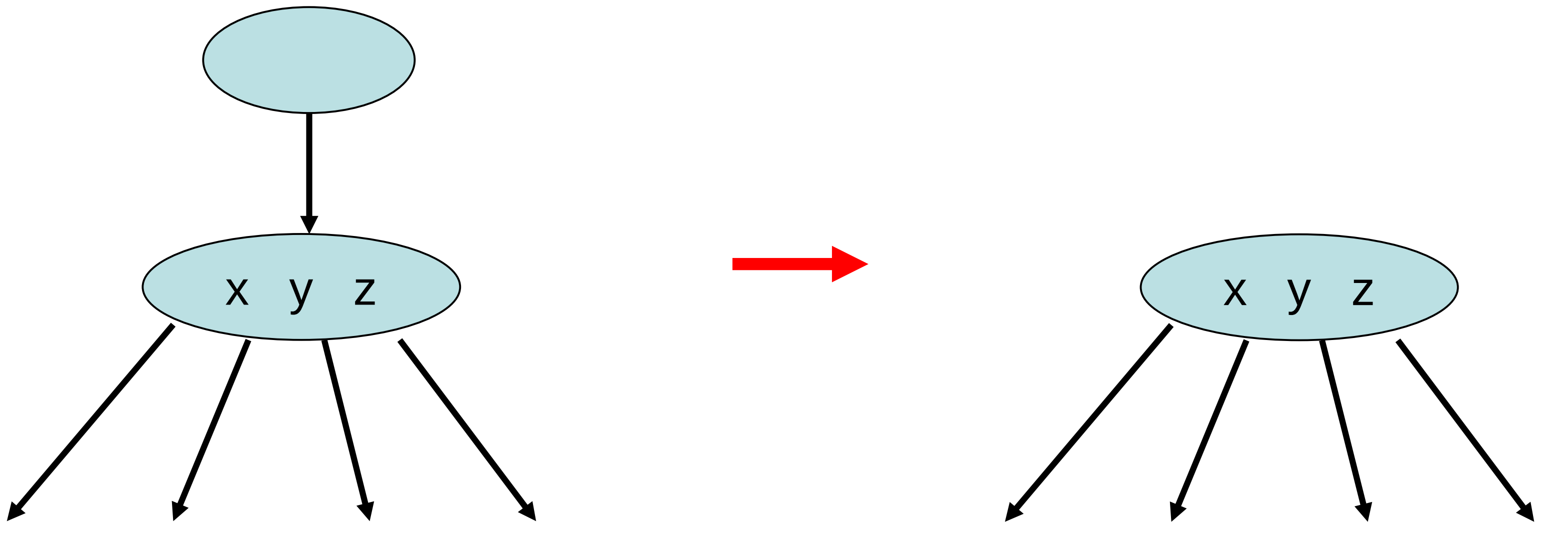
Delete(k) Operation

- If d(v)<a and the preceeding and succeeding siblings of v have degree a, merge v with one of these. (Example: a=3, b=5)



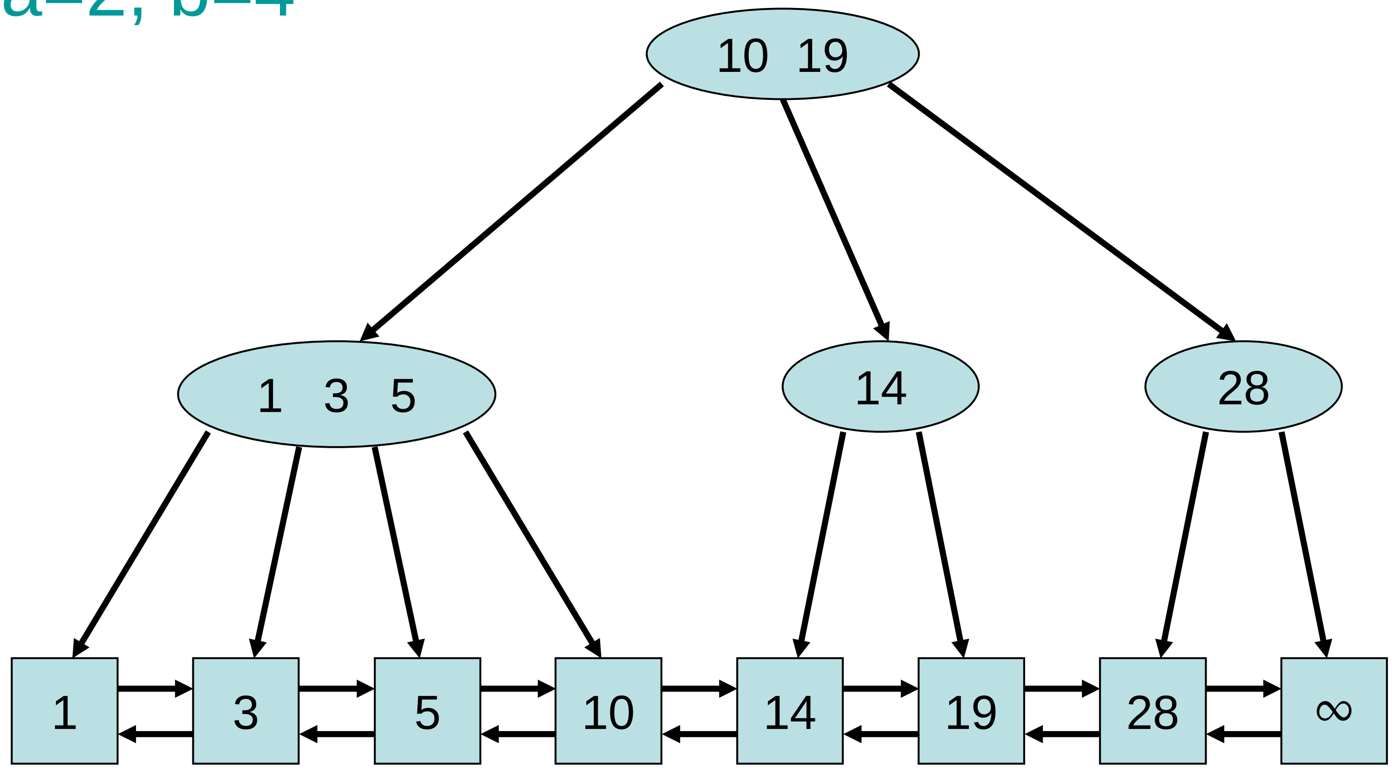
Delete(k) Operation

- Peform changes upwards until all inner nodes (except for the root) have degree $\geq a$. If root has degree <2: remove root.



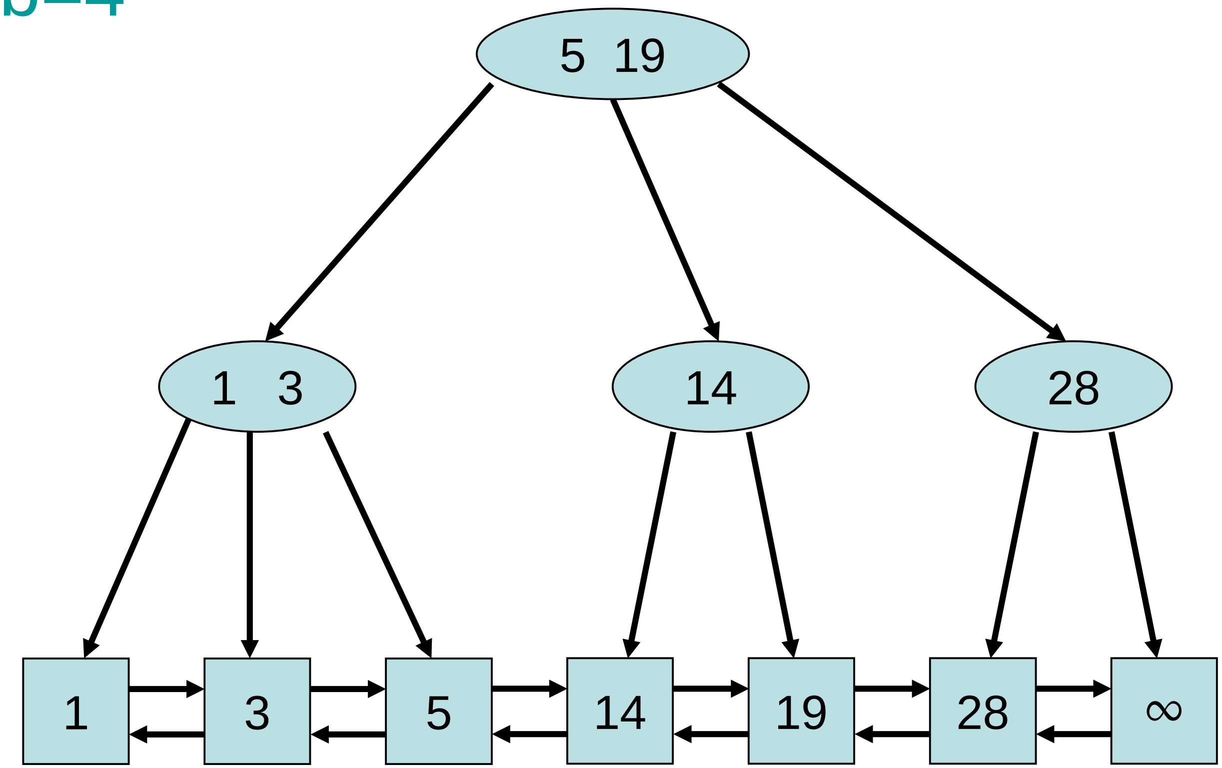
Delete(10)

a=2, b=4



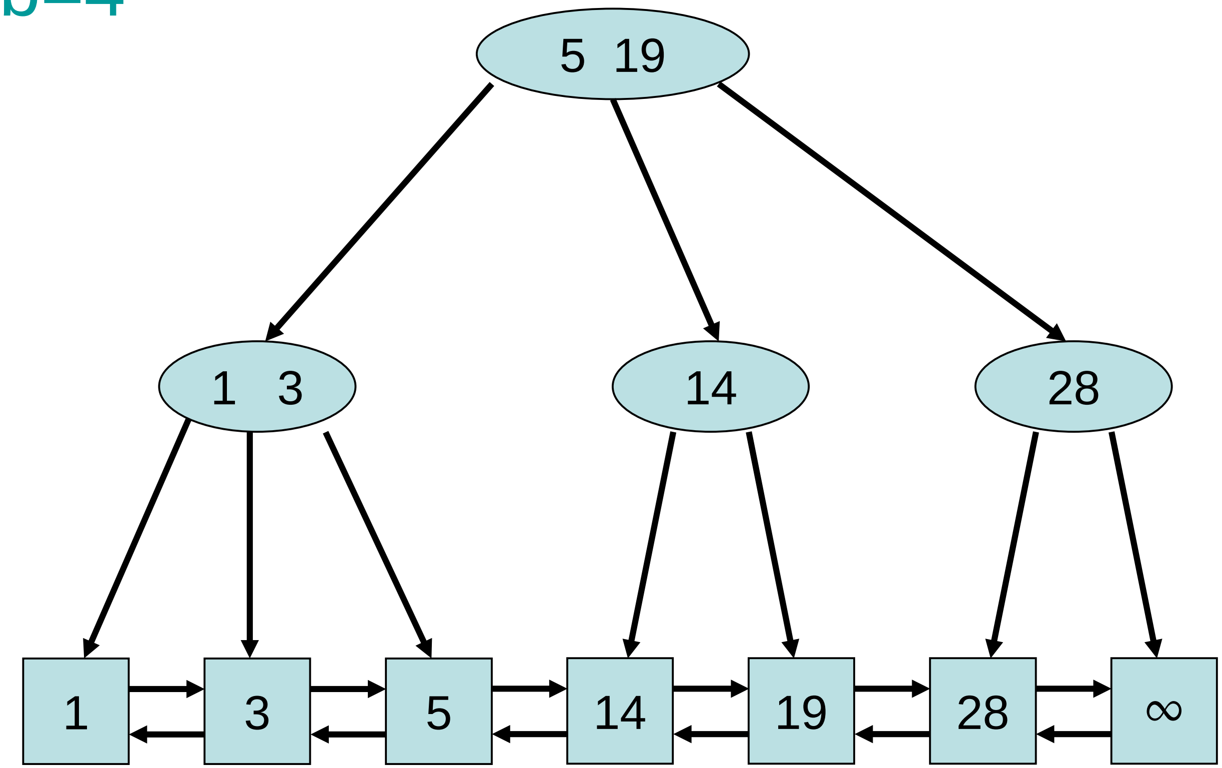
Delete(10)

a=2, b=4



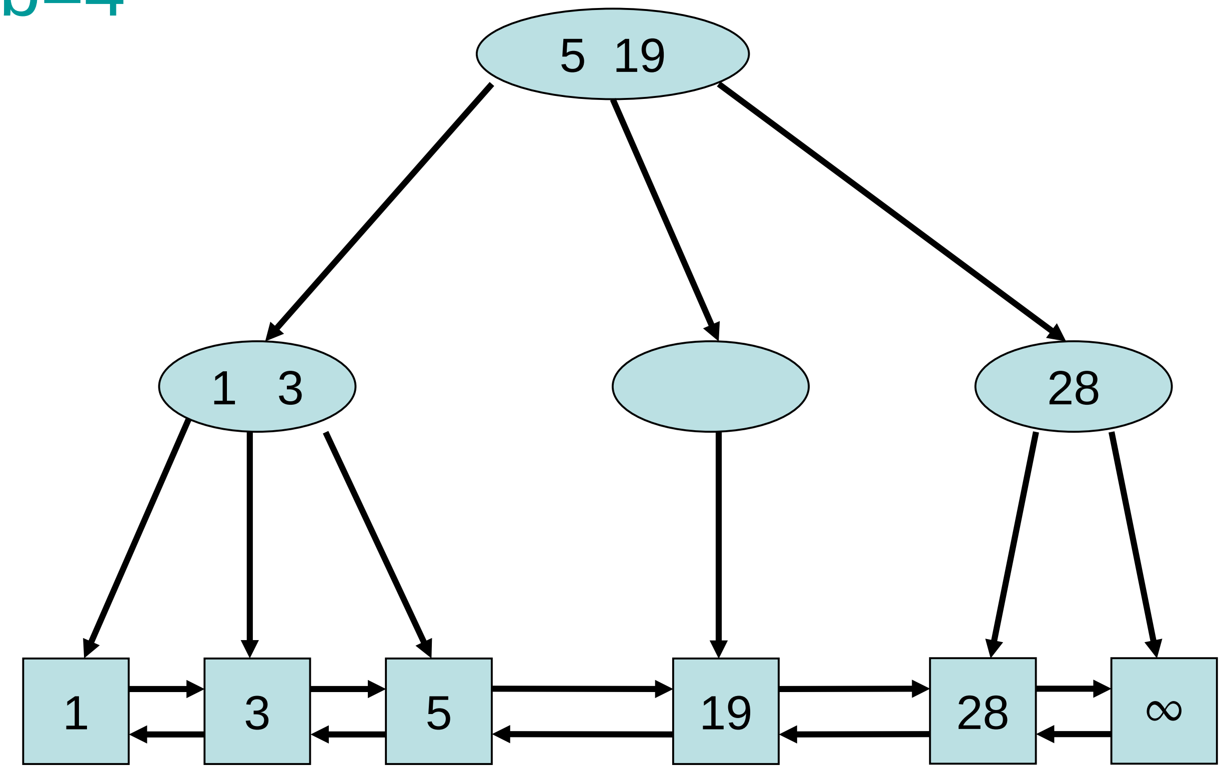
Delete(14)

a=2, b=4



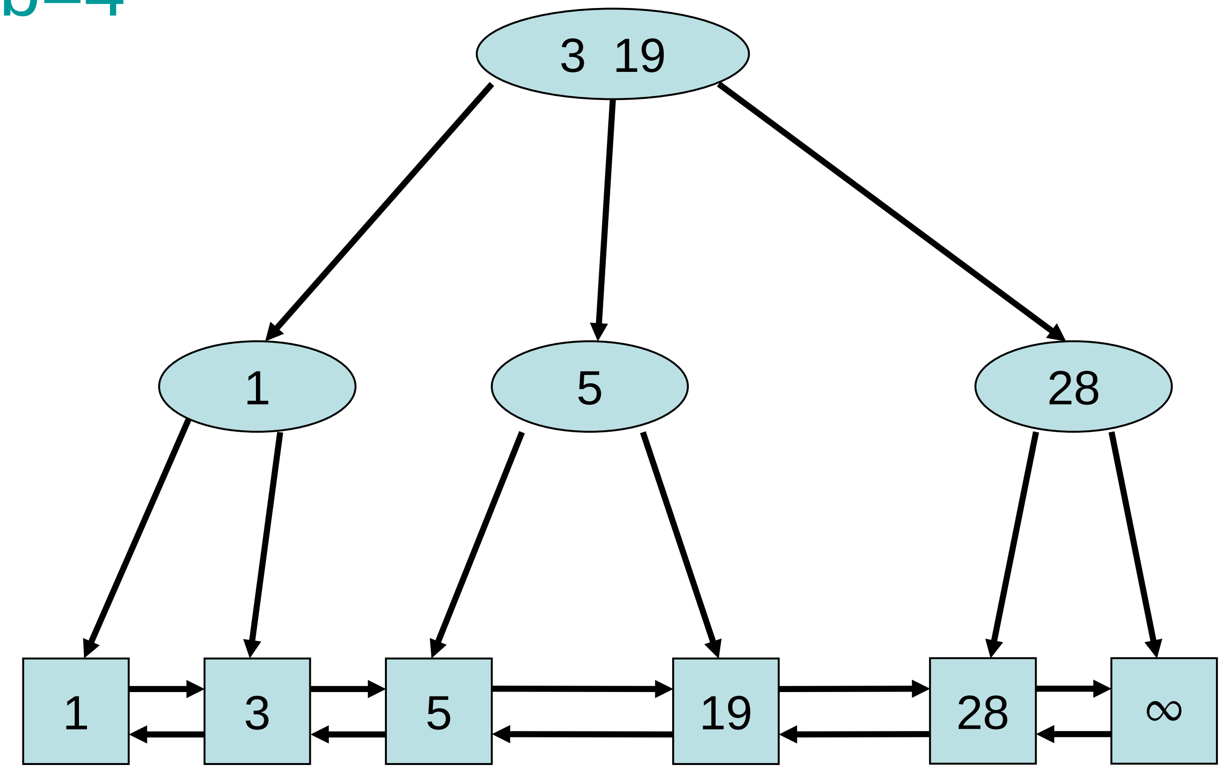
Delete(14)

a=2, b=4



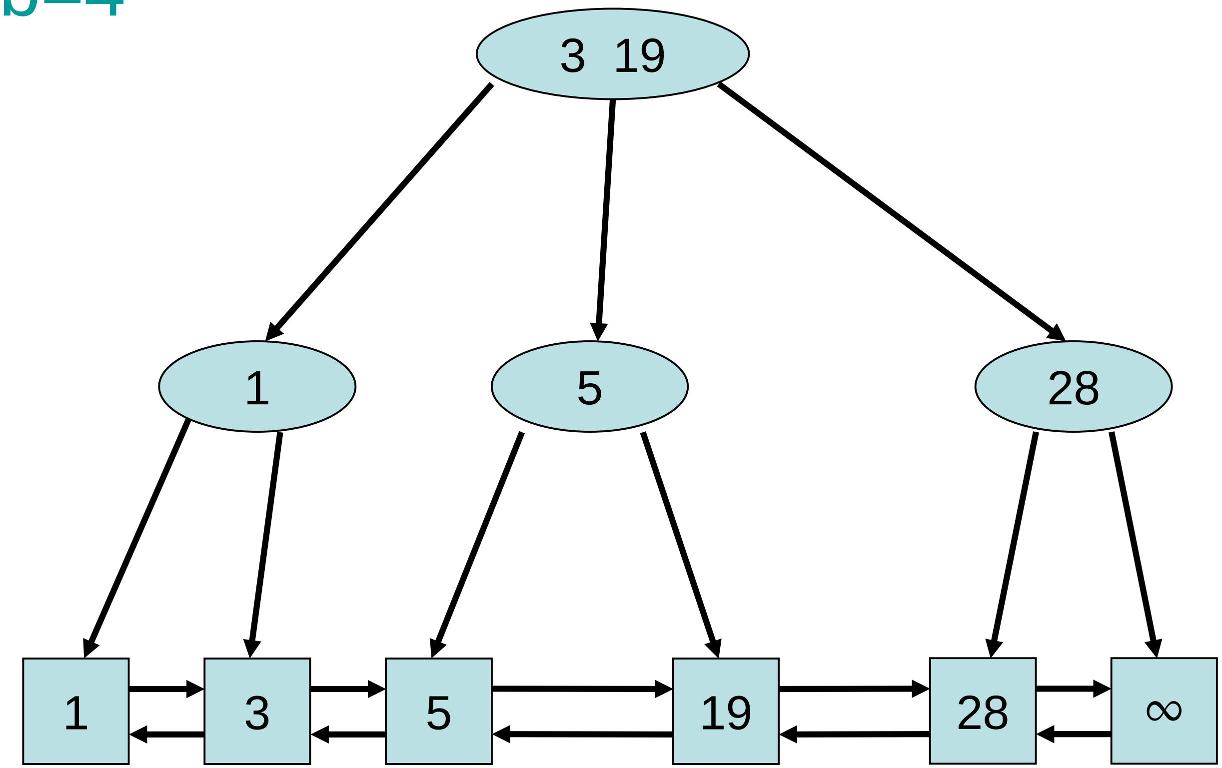
Delete(14)

a=2, b=4



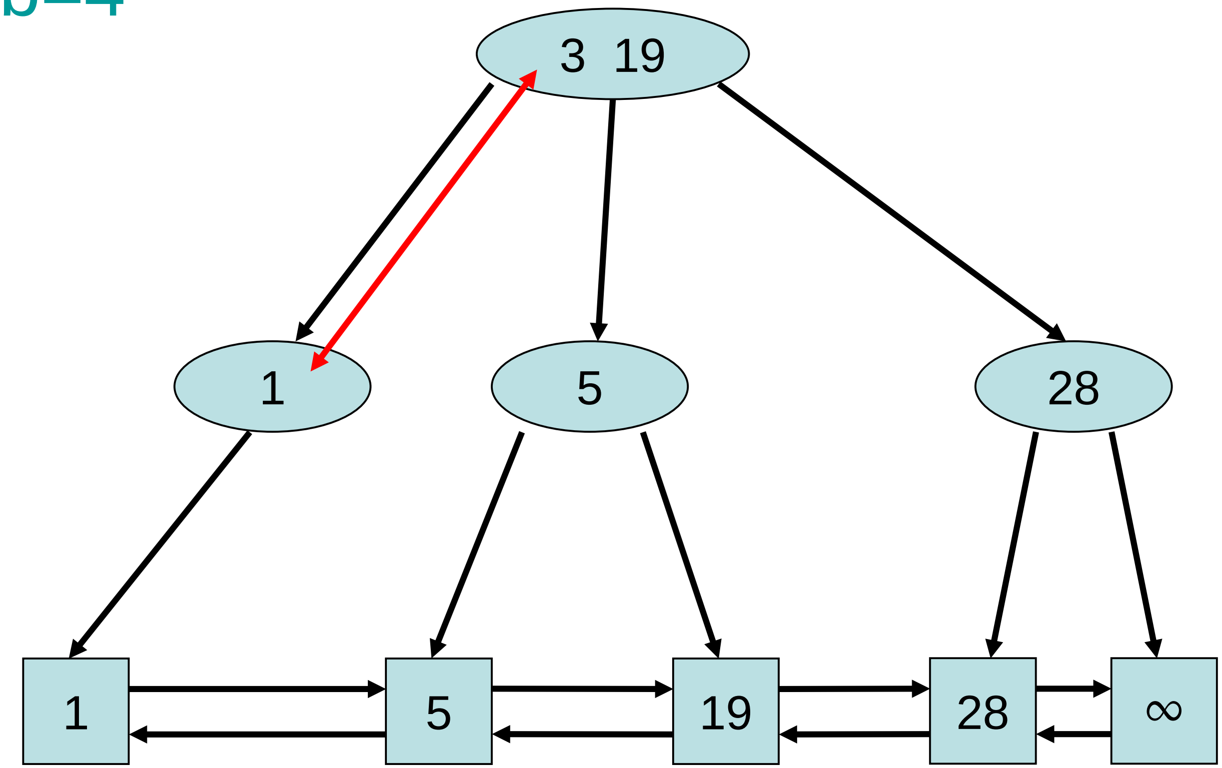
Delete(3)

a=2, b=4



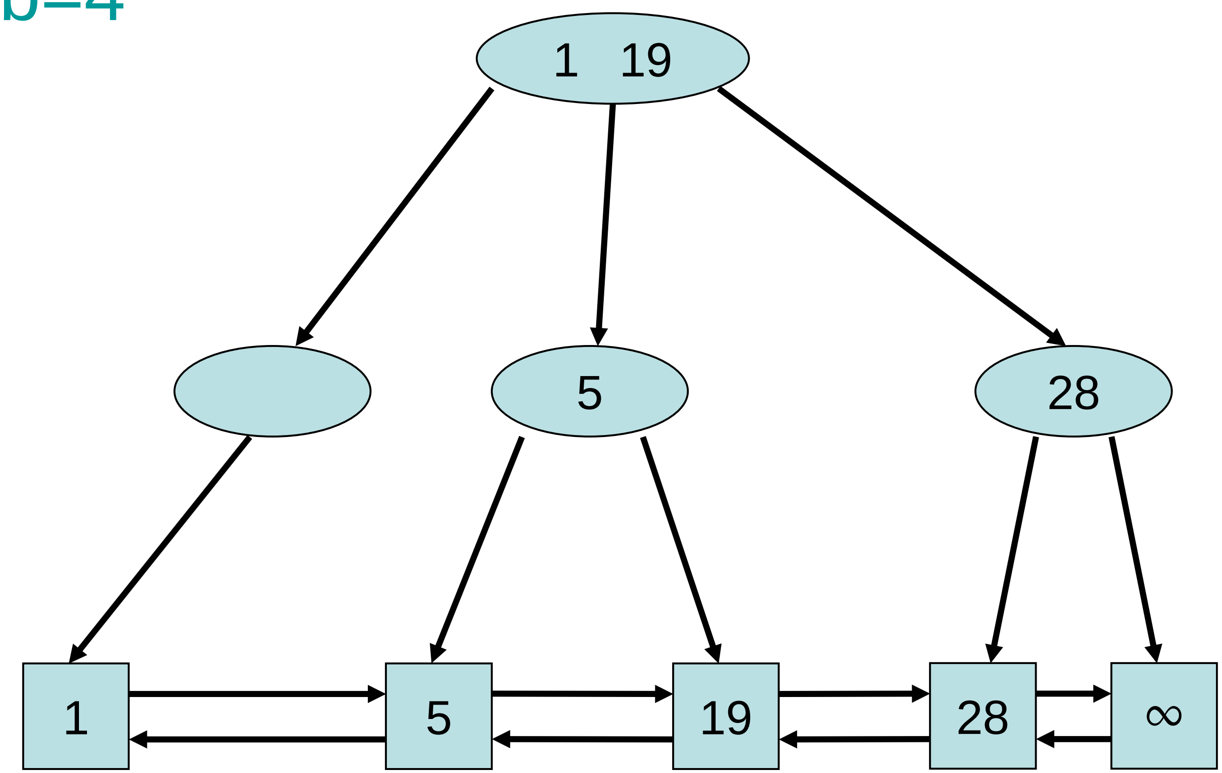
Delete(3)

a=2, b=4



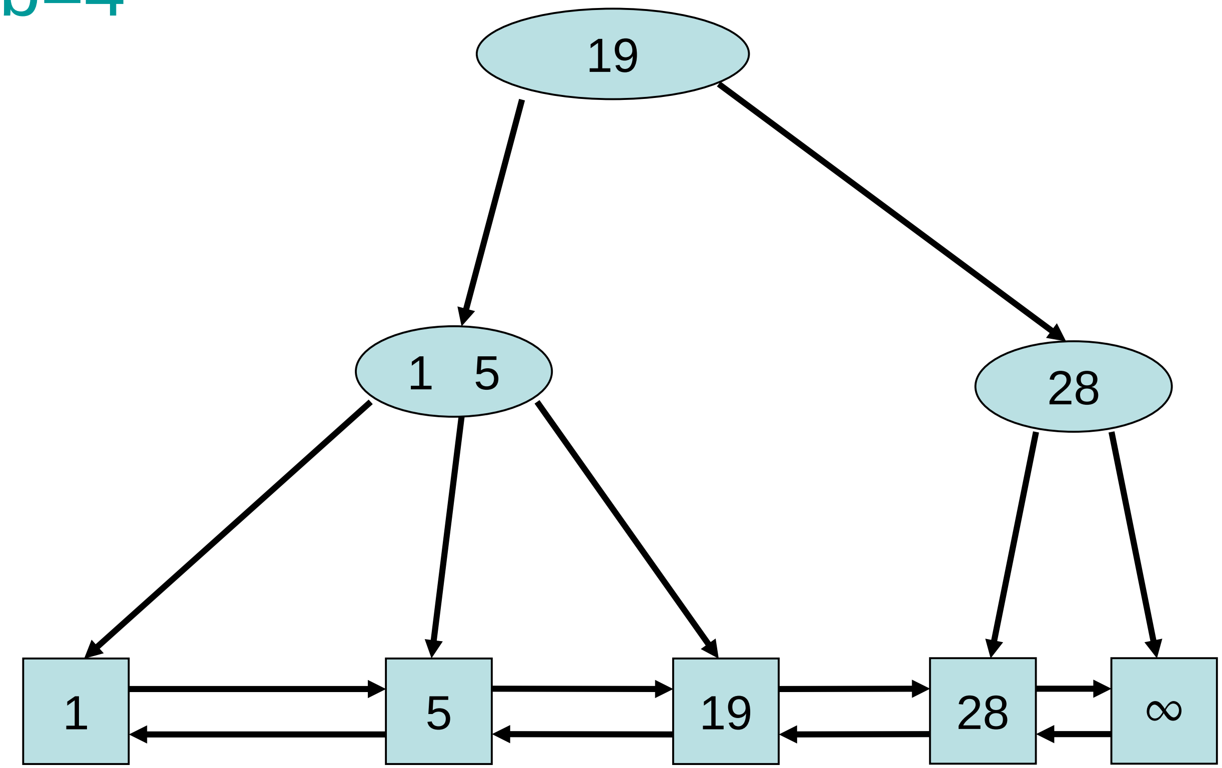
Delete(3)

a=2, b=4



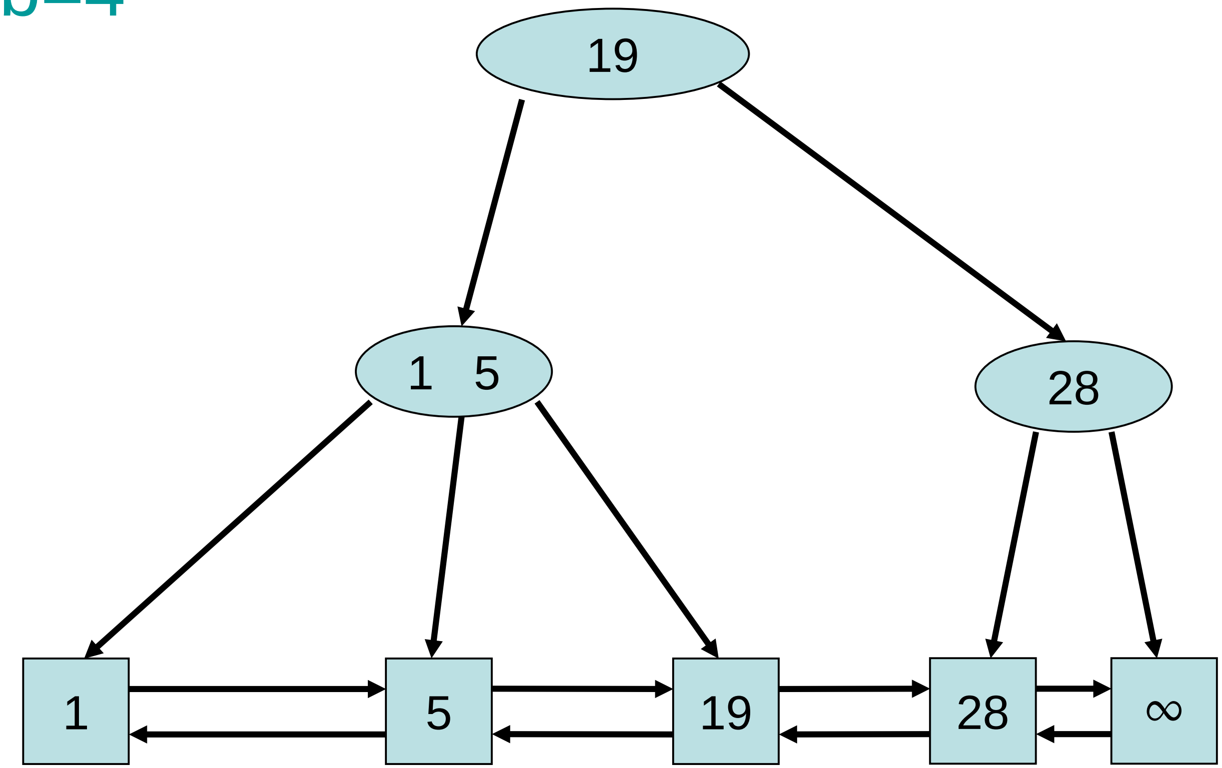
Delete(3)

a=2, b=4



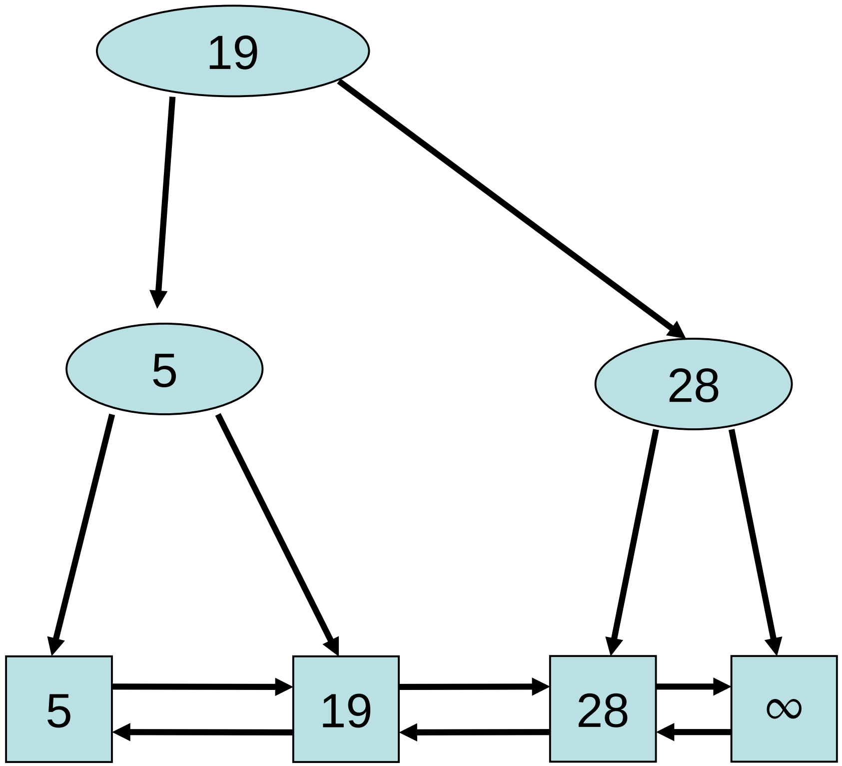
Delete(1)

a=2, b=4



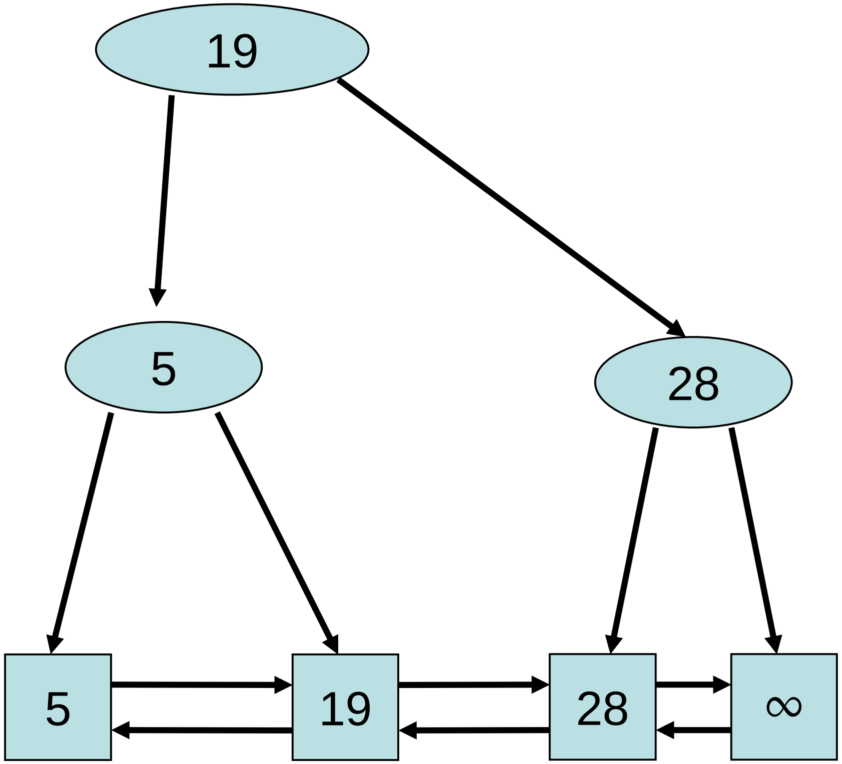
Delete(1)

a=2, b=4



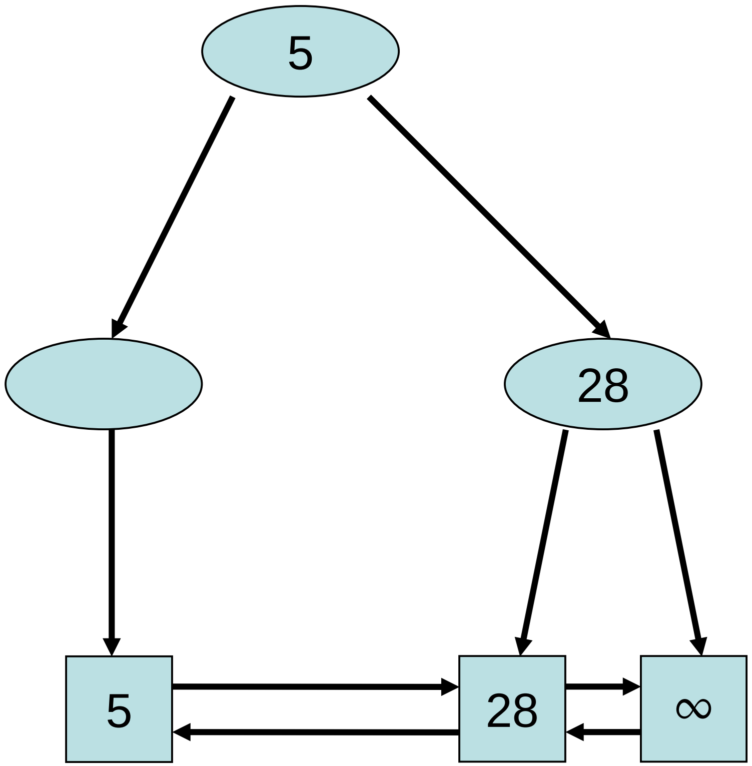
Delete(19)

a=2, b=4



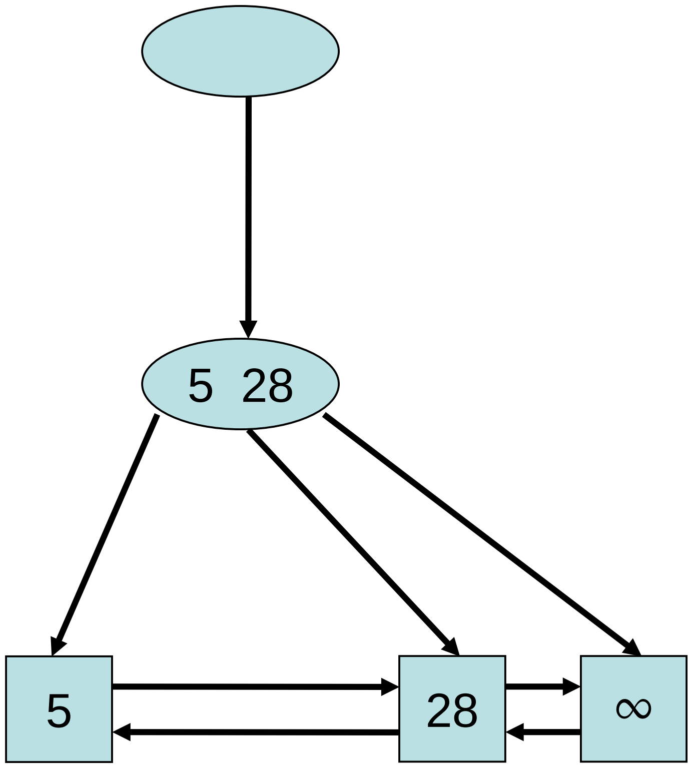
Delete(19)

a=2, b=4



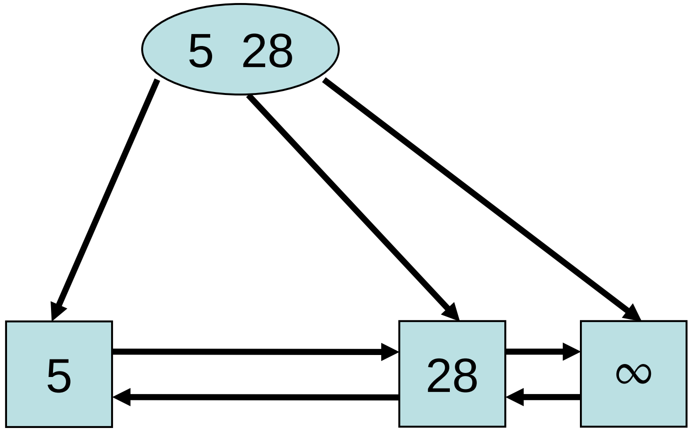
Delete(19)

a=2, b=4



Delete(19)

a=2, b=4



Delete Operation

- Form Invariant:
  For all leaves v,w: $t(v)=t(w)$
  Satisfied by Delete!

- Degree Invariant:
  For all inner nodes v except for the root: $d(v)\in[a,b]$, for root r: $d(r)\in[2,b]$

1) Delete merges node of degree $a-1$ with node of degree a. Since $b\geq 2a-1$, the resulting node has degree at most b.

2) Delete moves edge from a node of degree $>a$ to a node of degree $a-1$. Also OK.

3) Root deleted: children have been merged, degree of the remaining child is $\geq a$ (and also $\leq b$), so also OK.

More Operations

- min/max Operation:
  Pointers to both ends of list: time O(1).

- Range queries:
  To obtain all elements in the range [x,y], perform search(x) and go through the list till an element >y is found.
  Time O(log n + size of output).

n Update Operations

Theorem 3.11: There is a sequence of n insert and delete operations in a (2,3)-tree that require $\Omega(n \log n)$ many split and merge Operations.

Proof: Exercise

n Update Operations

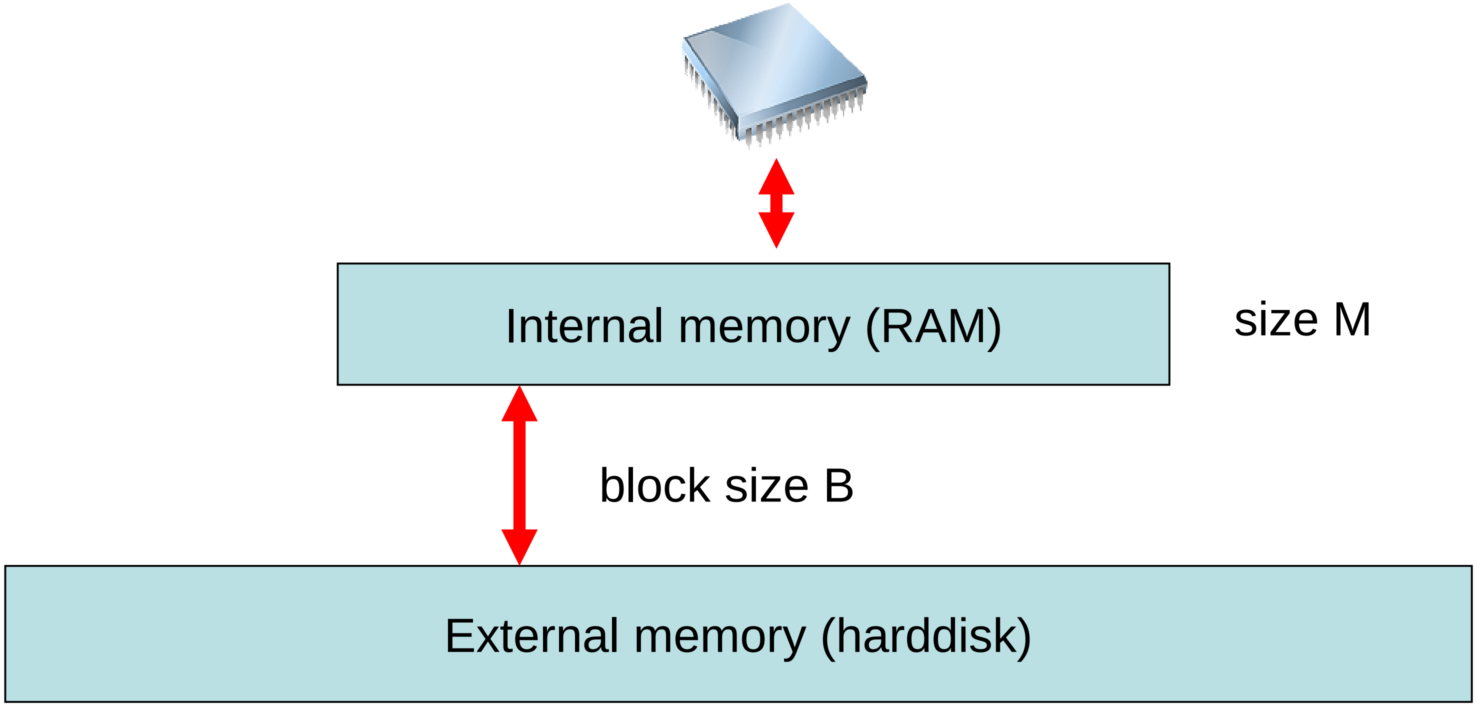
Theorem 3.12: Consider an (a,b)-tree with $b \geq 2a$ that is initially empty. For any sequence of n insert and delete operations, only O(n) split and merge operations are needed.

Proof:

Amortized analysis

External (a,b)-Tree

(a,b)-trees well suited for large amounts of data



External (a,b)-Tree

Problem: minimize number of block transfers between internal and external memory

Solution:

- use $b=B$ (block size) and $a=b/2$
- keep highest $(1/2)\cdot\log_a(M/b)$ levels of (a,b)-tree in internal memory (storage needed $\leq M$)
- Lemma 3.10: depth of (a,b)-tree $\leq 1+\lfloor\log_a (n+1)/2\rfloor$
- $\log_a[(n+1)/2] - (1/2)\cdot\log_a(M/b) \leq \log_a[(n+1)/(2\sqrt{M})] + \log_a b$
- $\log_a b = O(1)$
- Cost for insert, delete and search operations: $O(\log_B(n/\sqrt{M}))$ block transfers

External (a,b)-Tree

Problem: minimize number of block transfers between internal and external memory

A better analysis can show (exercise):

- Cost for insert, delete and search operations: $\sim 2\log_{B/2}(n/M)+1$ block transfers ($+1$: list access)

Example:

- $n = 100{,}000{,}000{,}000{,}000$ keys
- $M = 16$ Gbyte (~4,000,000,000 keys)
- $B = 256$ Kbyte (~64,000 keys)
- $2\log_{B/2}(n/M)+1 \leq 3$

Search Trees

Problem: binary tree can degenerate!

Solutions:

- Splay tree
  (very effective heuristic)
- (a,b)-tree
  (guaranteed well balanced)
- hashed Patricia trie
  (loglog-search time)

Applications

Longest Prefix Search

- All keys are encoded as binary sequence $\{0,1\}^W$
- Prefix of a key $x \in \{0,1\}^W$: arbitrary subsequence of x that starts with the first bit of x (example: 101 is a prefix of 10110100)

Problem: given a key $x \in \{0,1\}^W$, find a key $y \in S$ with longest common prefix

Solution: Trie Hashing

Trie

A trie is a search tree over some alphabet Σ that has the following properties:

- Every edge is associated with a symbol $c \in \Sigma$
- Every key $x \in \Sigma^k$ that has been inserted into the trie can be reached from the root of the trie by following the unique path of length k whose edge labels result in x.

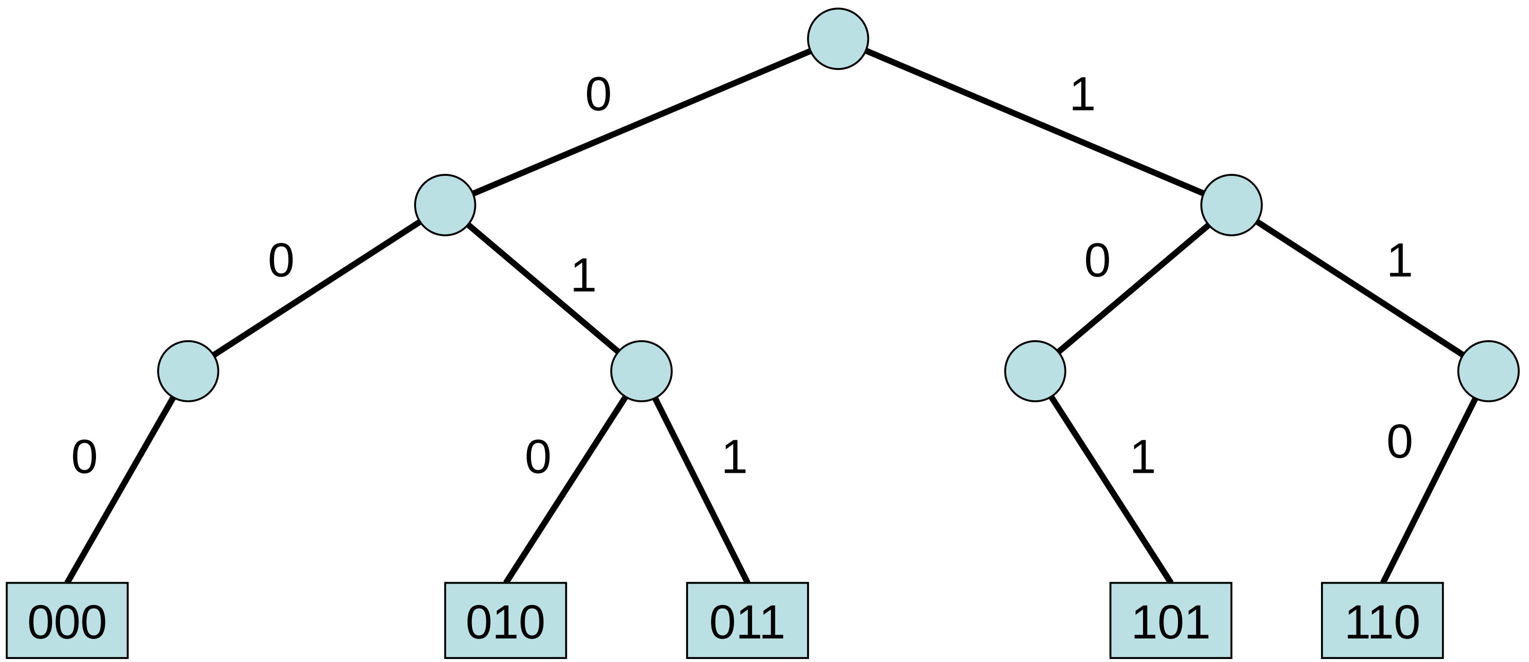
For simplicity: all keys from $\{0,1\}^W$ for some $W \in \mathbb{N}$.

Example:

$(0,2,3,5,6)$ with $W=3$ results in $(000,010,011,101,110)$

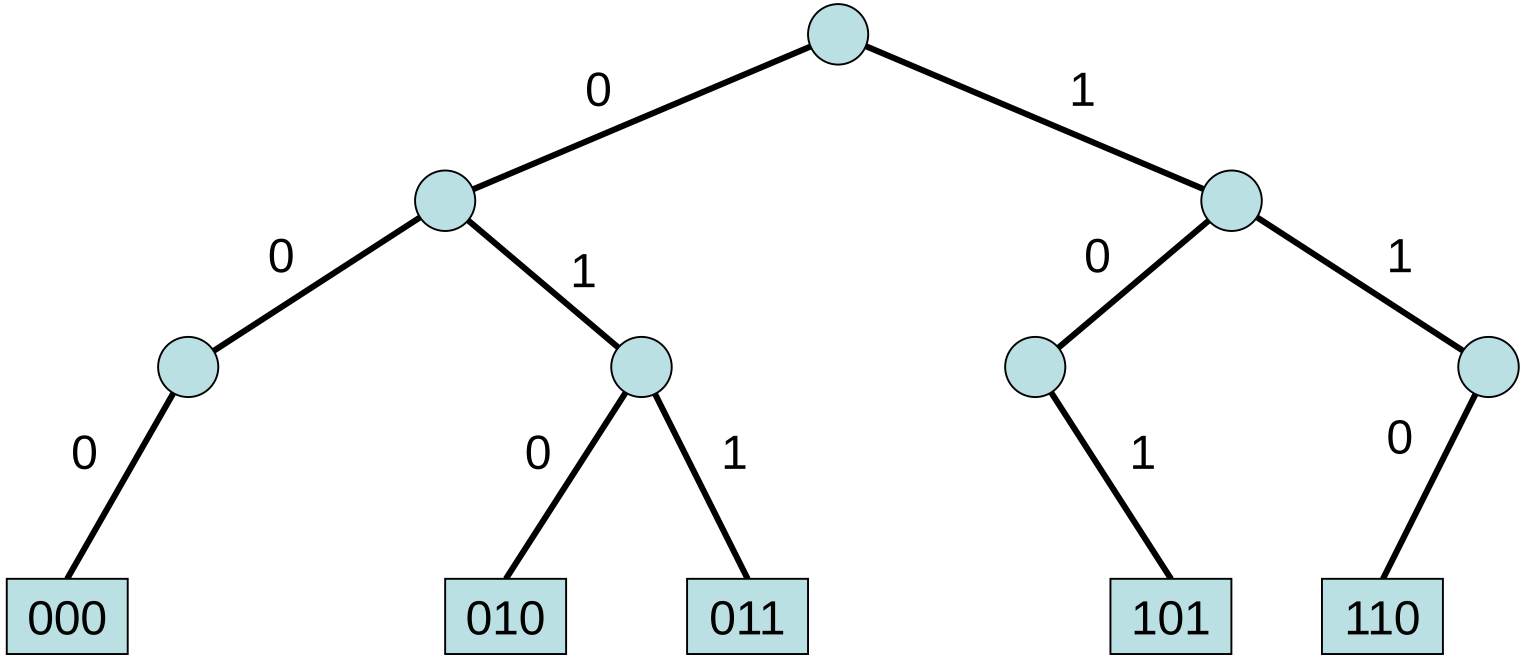
Trie

Example: (without list at bottom)



Trie

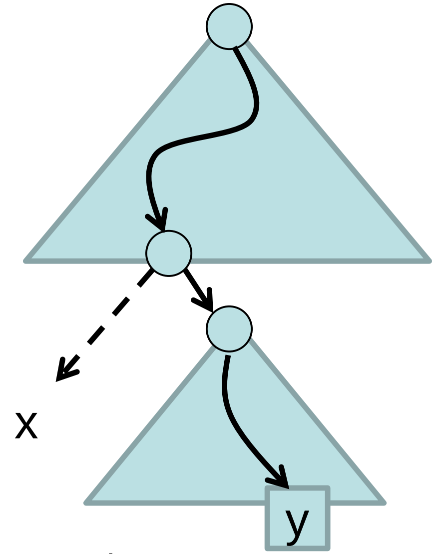
search(4) (4 corresponds to 100):



Output: 5 (longest common prefix)

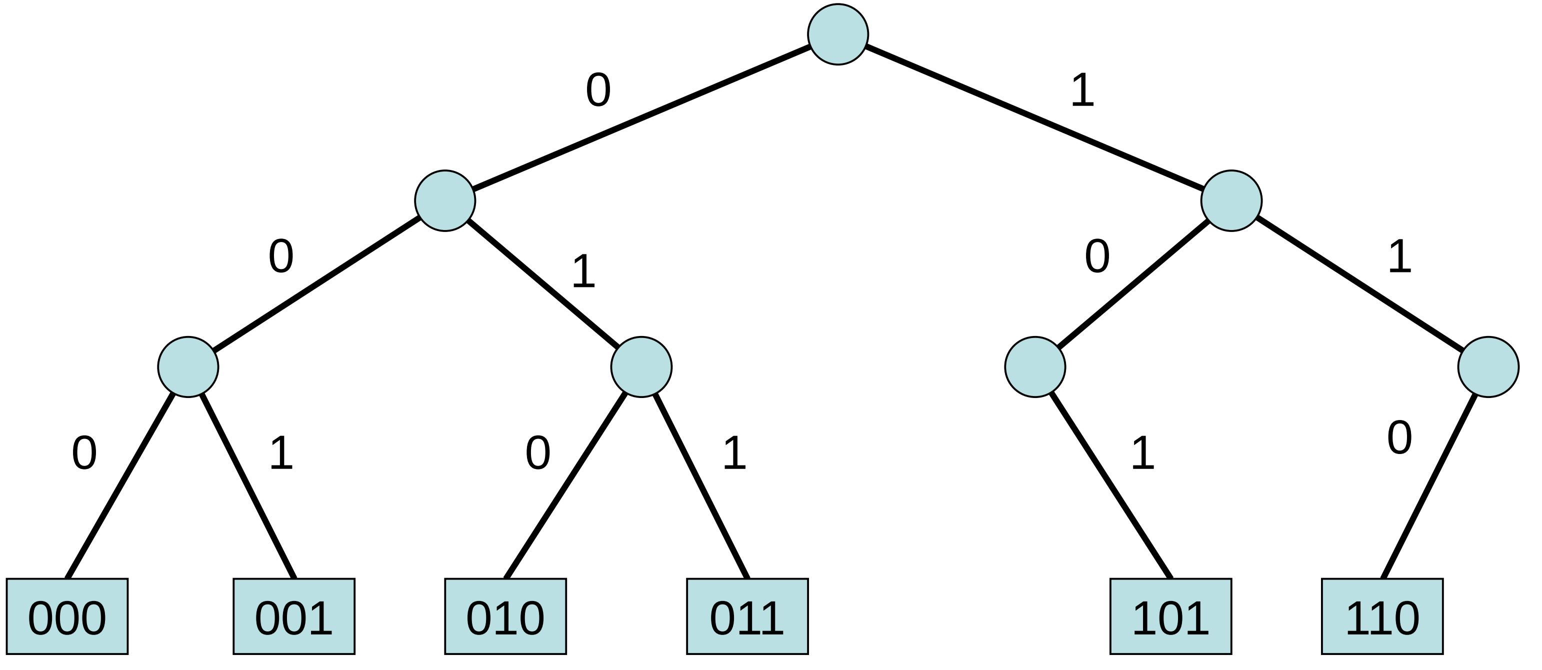
Trie

In general: a search(x) request follows the edges in the trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, the request may be forwarded to any leaf y in the subtrie below since all of them have the same longest prefix match with x.



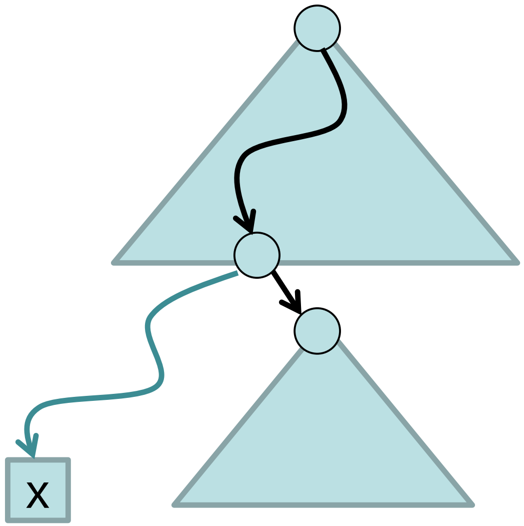
Trie

insert(1) (1 corresponds to 001):



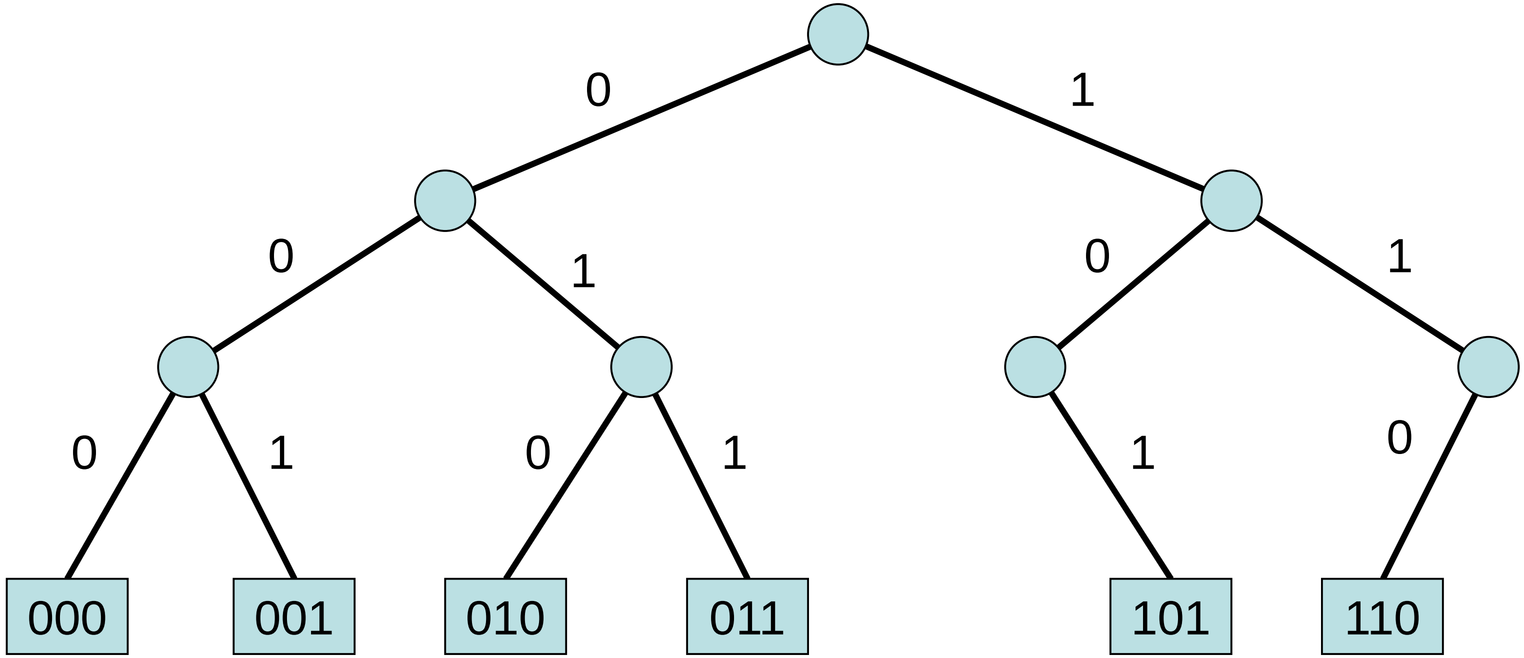
Trie

In general: an insert(x) request follows the edges in the trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, a new path (of length the remaining bits in x) is created that leads to the new leaf x.



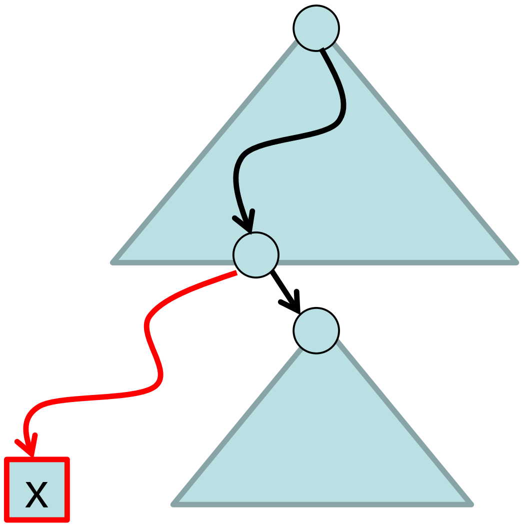
Trie

delete(5):



Trie

In general: a delete(x) request follows the edges in the trie down to the leaf x. If x does not exist, the delete operation terminates. Otherwise, x as well as the chain of nodes upwards till the first node with at least two children is deleted.



Patricia Trie

Problem:

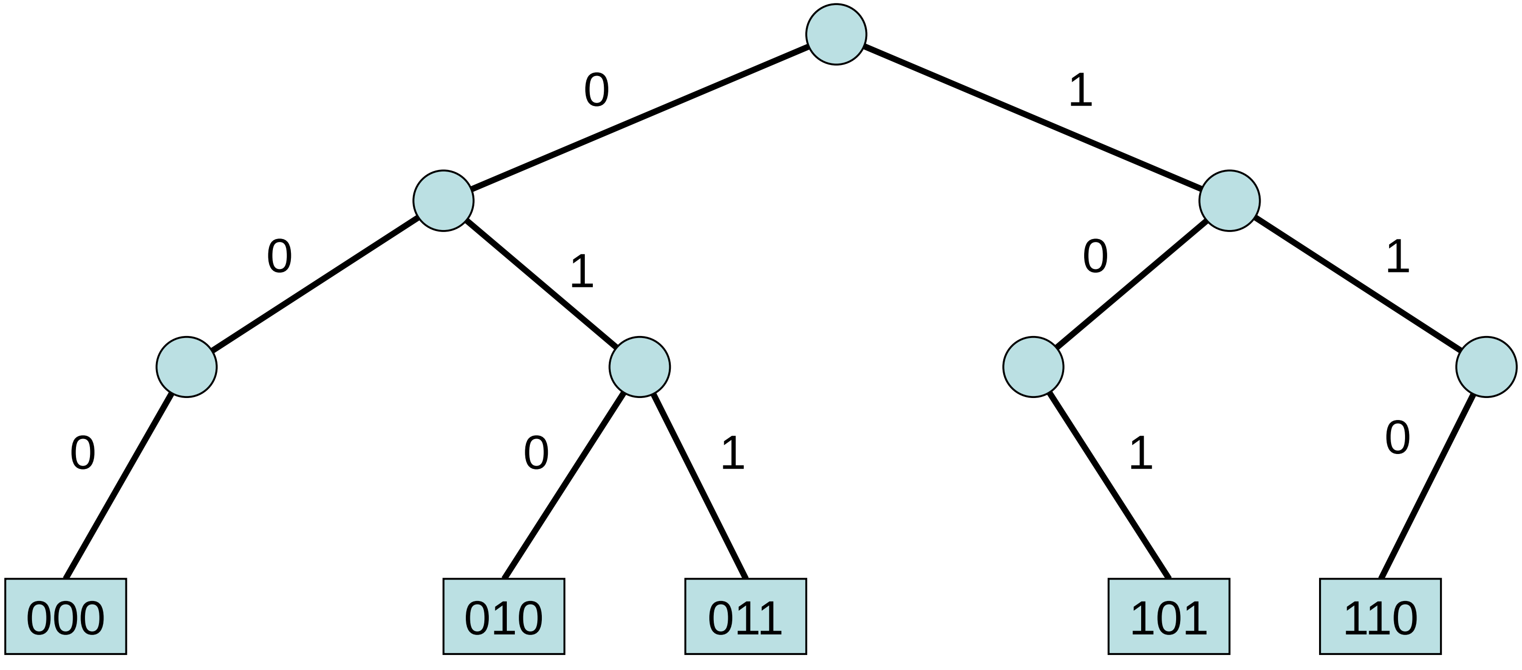
- Longest common prefix search for some $x \in \{0,1\}^W$ can take $\Theta(W)$ time.
- Insert and delete may require $\Theta(W)$ structural changes in the trie.

Improvement: use Patricia trie

A Patricia trie is a compressed trie in which all chains (i.e., maximal sequences of nodes of degree 1) are merged into a single edge whose label is equal to the concatenation of the labels of the merged trie edges.

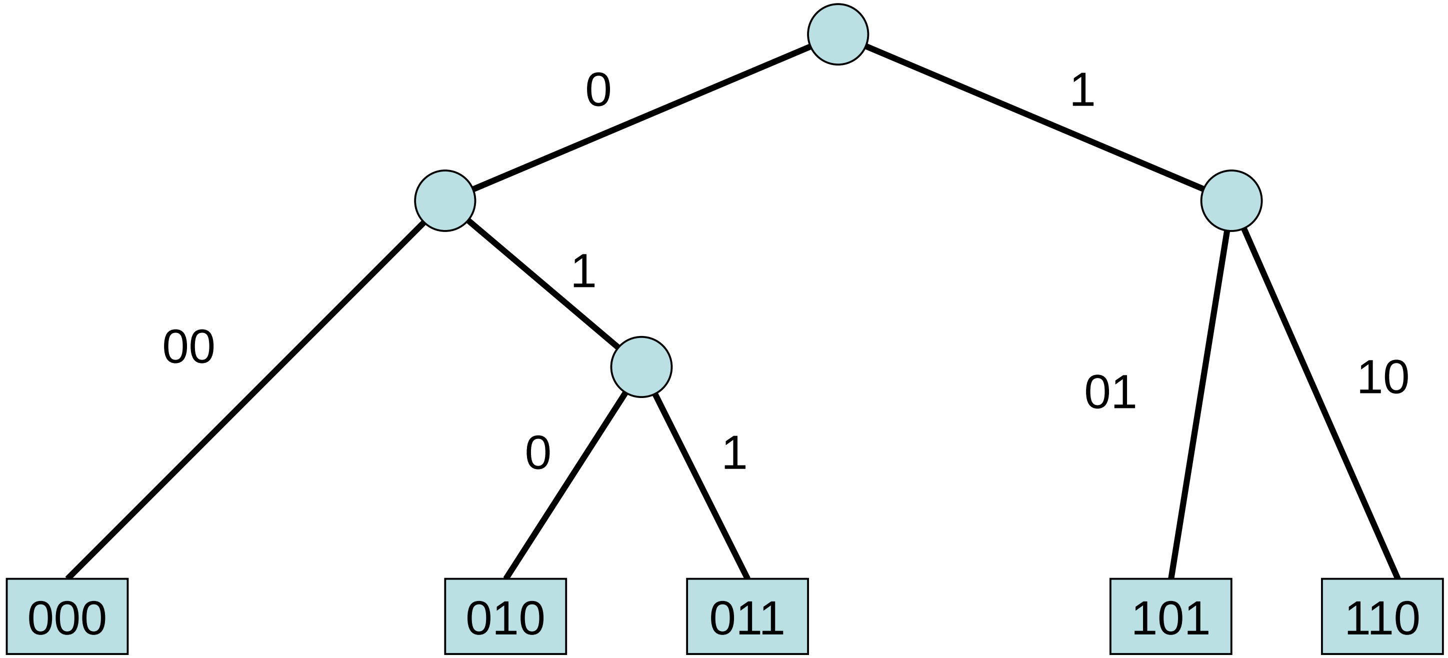
Trie

Example 1:



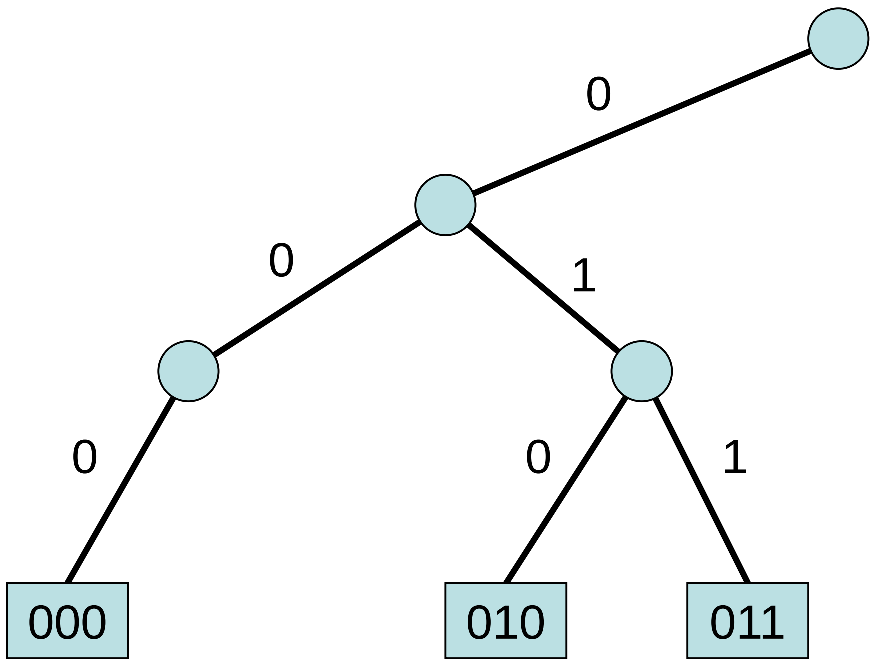
Patricia Trie

Example 1:



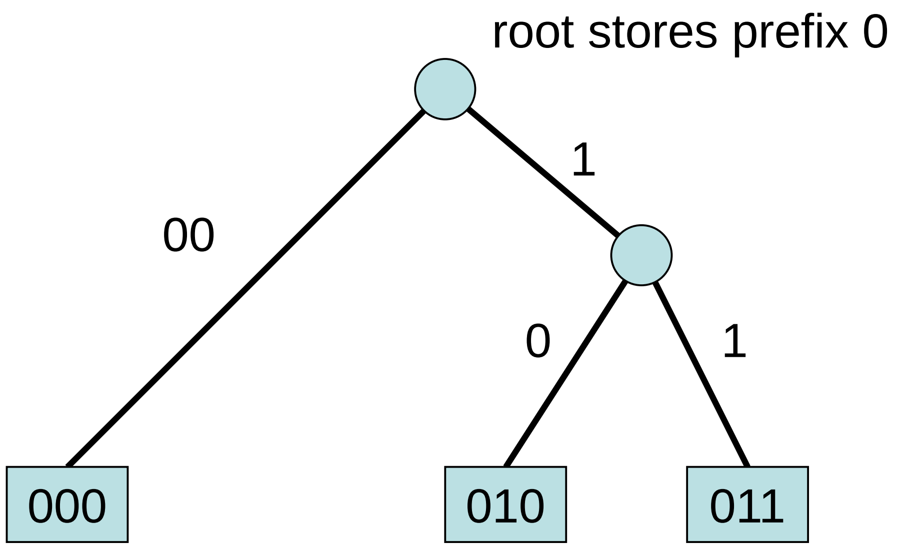
Trie

Example 2:



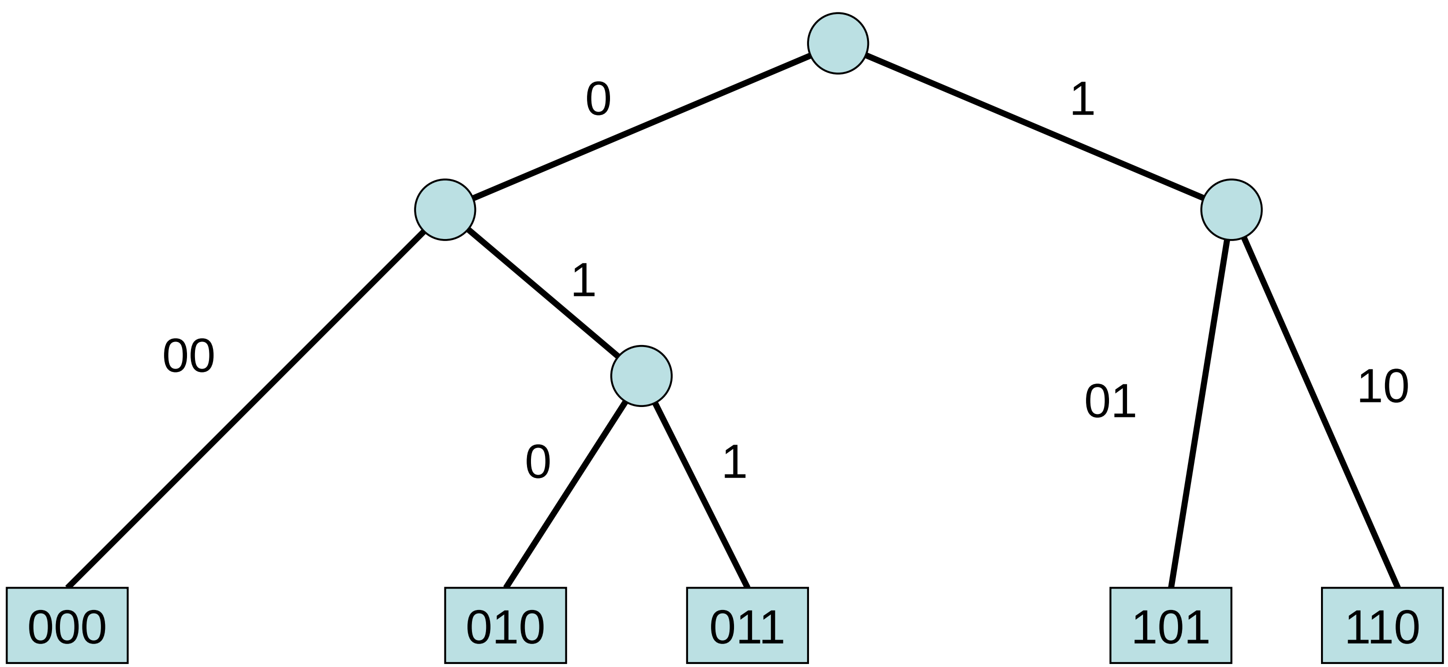
Patricia Trie

Example 2:



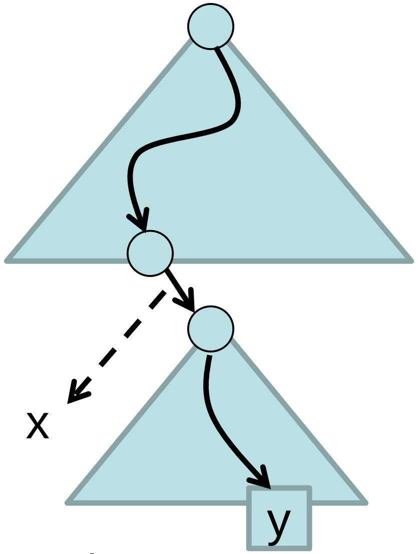
Patricia Trie

search(4):



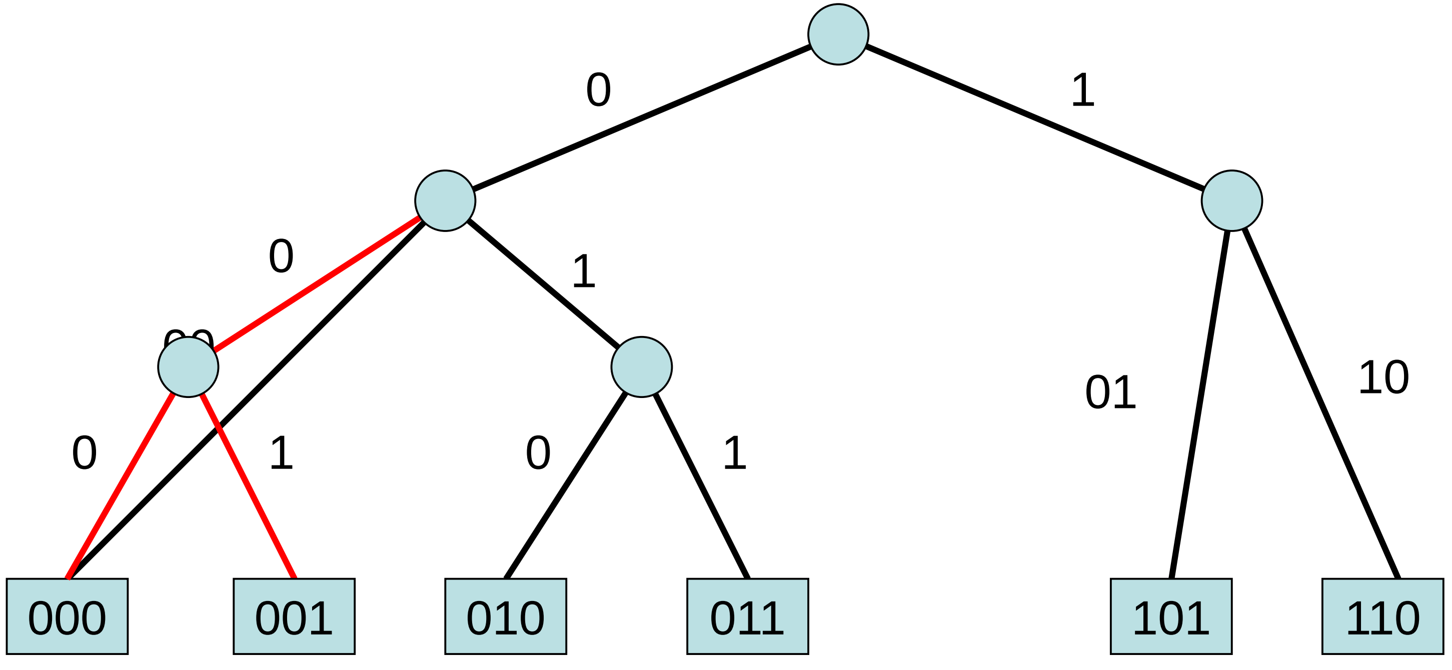
Patricia Trie

In general: a search(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once no edge is available any more to follow the bits in x, the request may be forwarded to any leaf y in the subtrie below since all of them have the same longest prefix match with x.



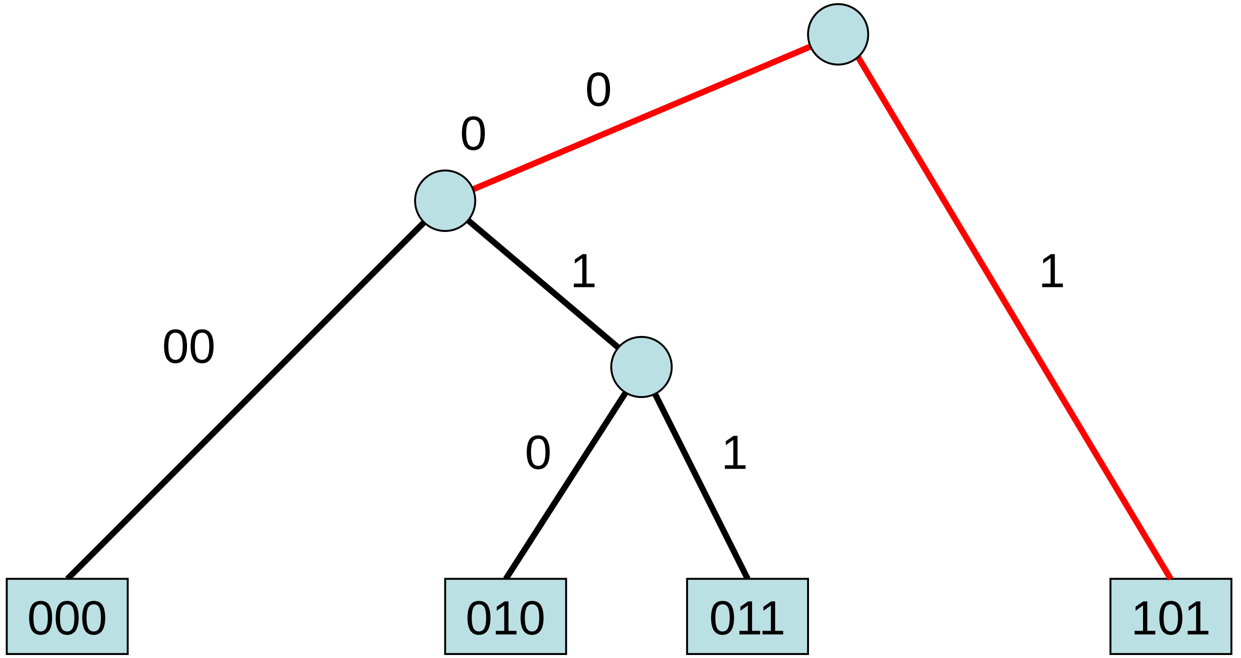
Patricia Trie

insert(1):



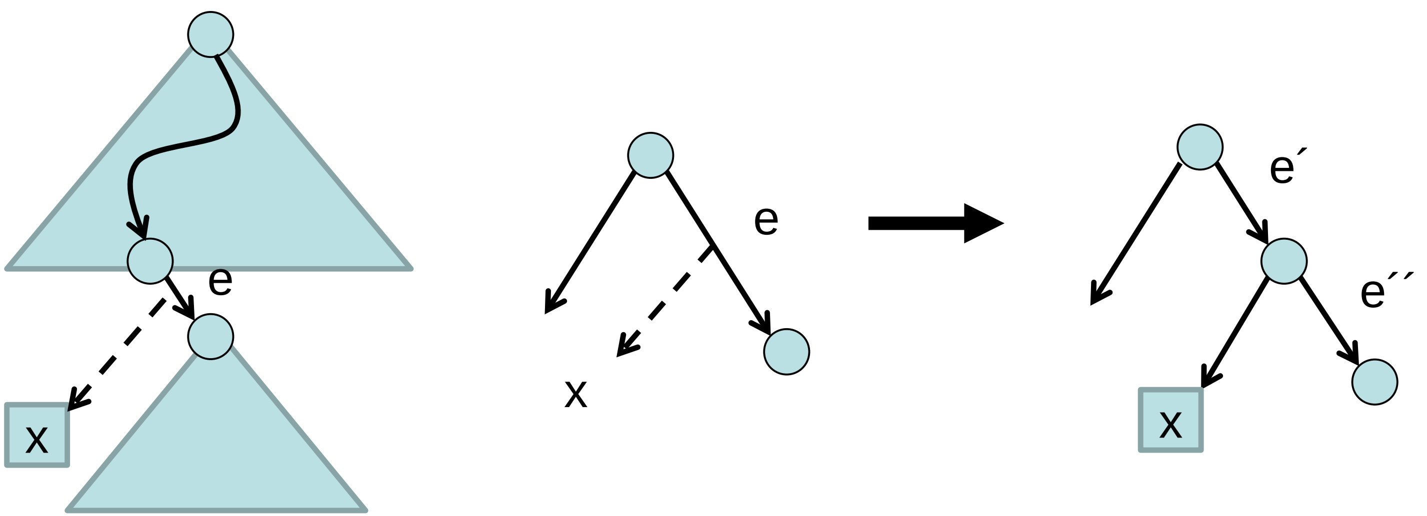
Patricia Trie

Insert(5):



Patricia Trie

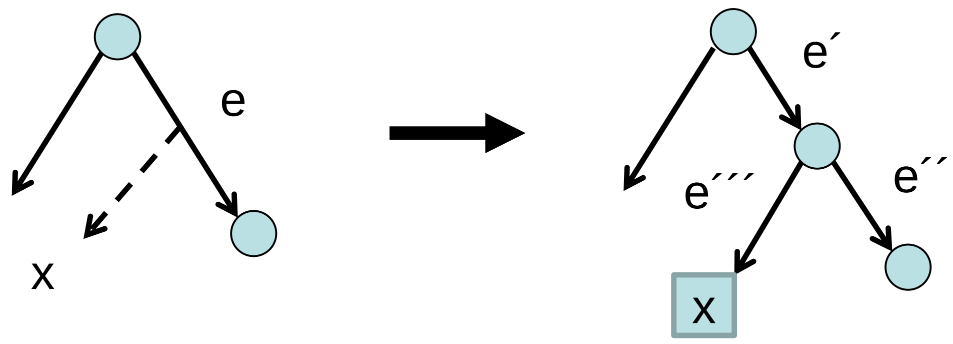
In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.



Patricia Trie

In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.

Example: l(e)=10010, x=…10110100

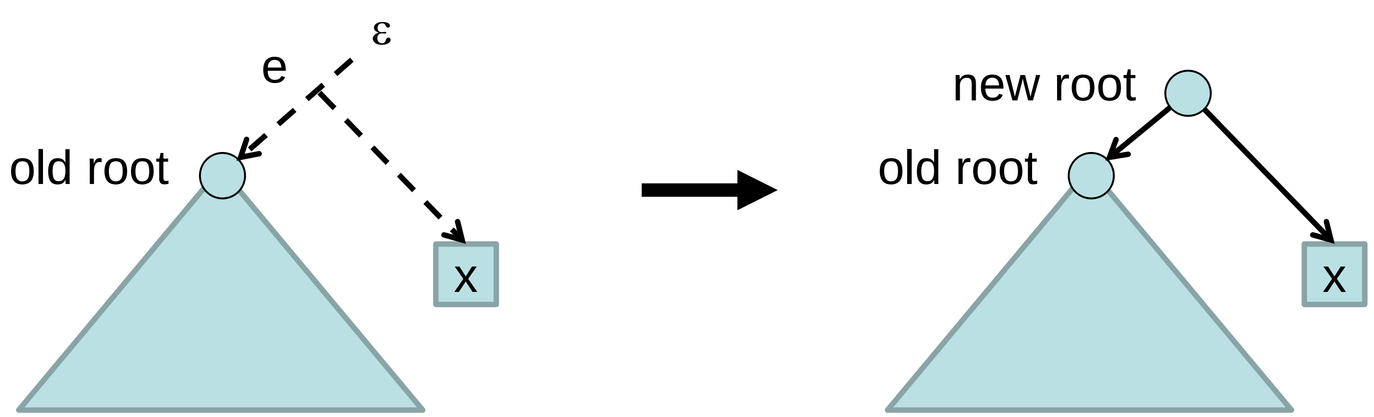


l(e´)=10
l(e´´)=010
l(e´´´)=110100

Patricia Trie

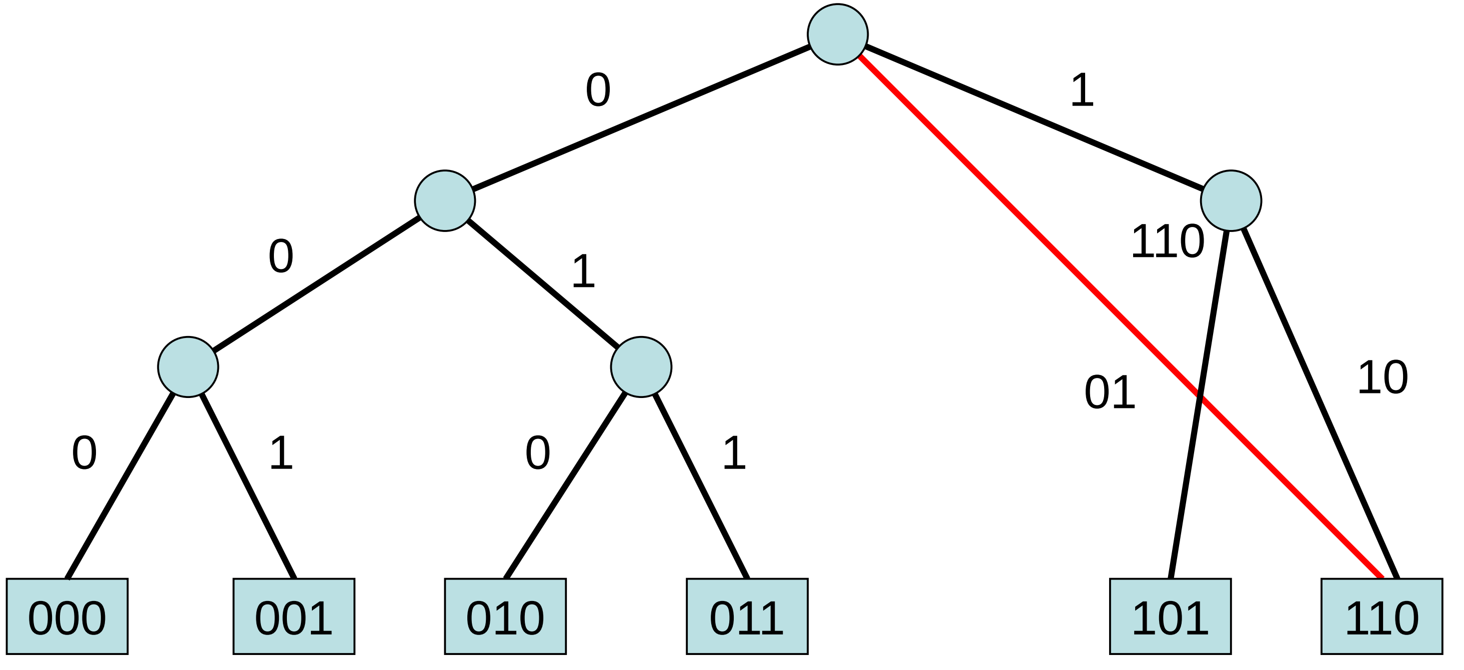
In general: an insert(x) request follows the edges in the Patricia trie as long as their labels form a prefix of x. Once an edge e is reached whose label l(e) does not follow the bits in x, a new tree node is created in the middle of e.

Special case:



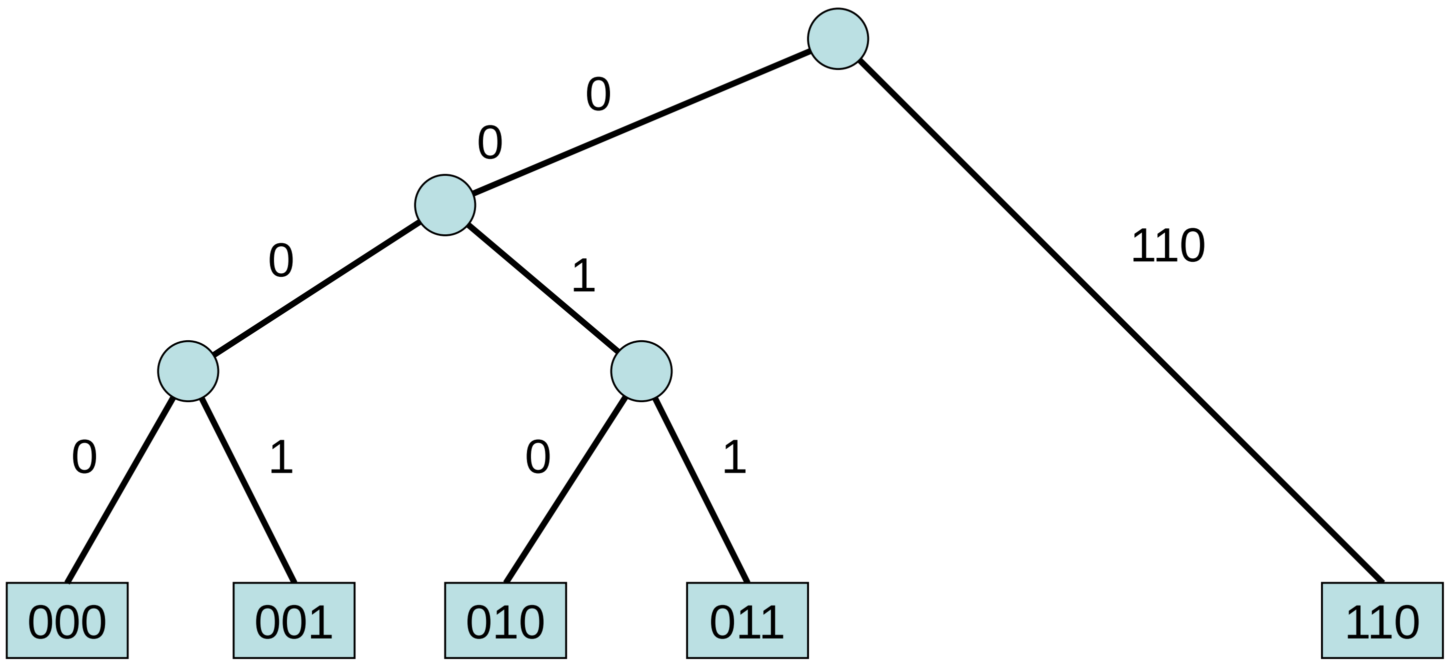
Patricia Trie

delete(5):



Patricia Trie

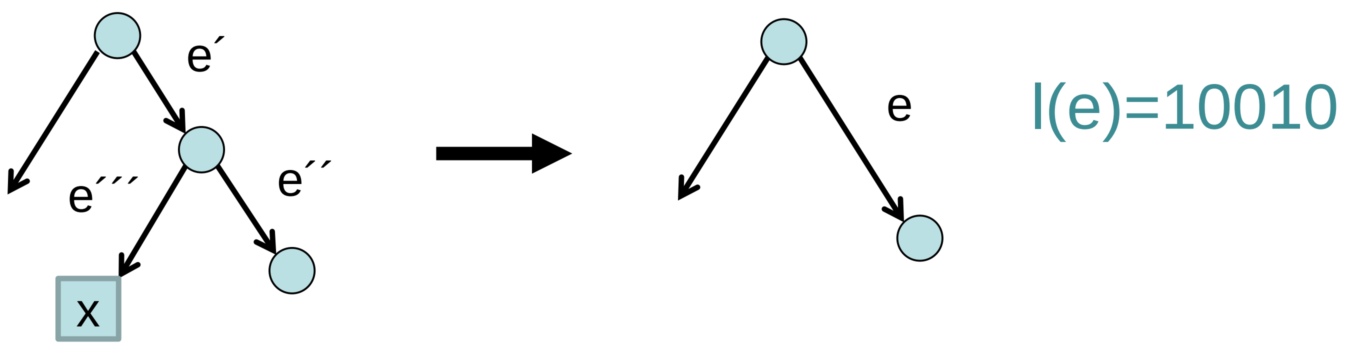
delete(6):



Patricia Trie

In general: a delete(x) request follows the edges in the Patricia trie down to the leaf x. If x does not exist, the delete operation terminates. Otherwise, x as well as its parent are deleted.

Example: l(e´)=10, l(e´´)=010, l(e´´´)=110100, x=…10110100



Patricia Trie

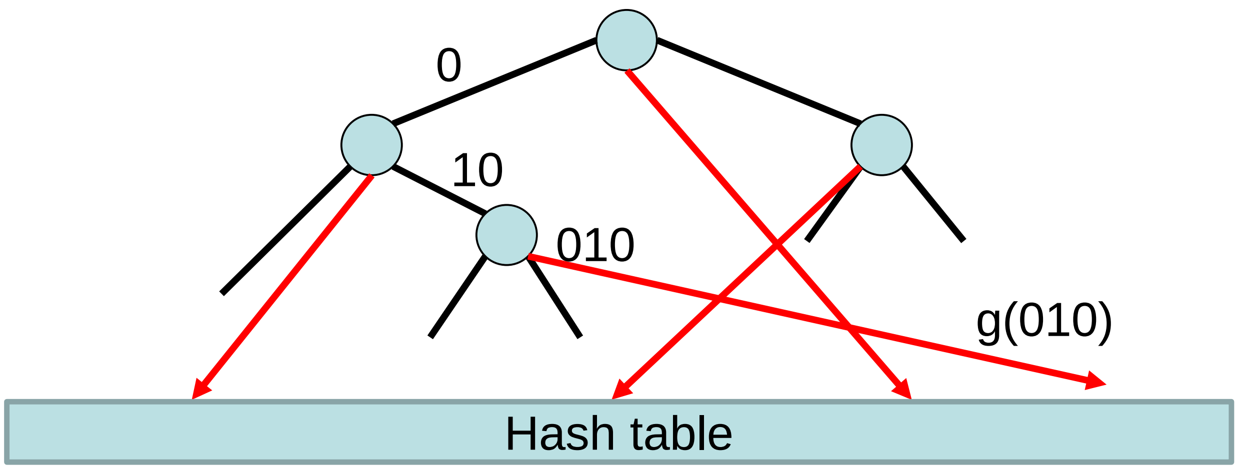
- Search, insert, and delete like in an ordinary binary tree, with the difference that we have labels at the edges.
- Search time still O(W) in the worst case, but just O(1) structural changes.

Idea: To improve search time, we hash the Patricia trie to some hash table (using, for example, cuckoo hashing).

Patricia Trie

Hashing to some hash table:

- Node label: concatenation of edge labels from root
- Every node is hashed according to its node label.

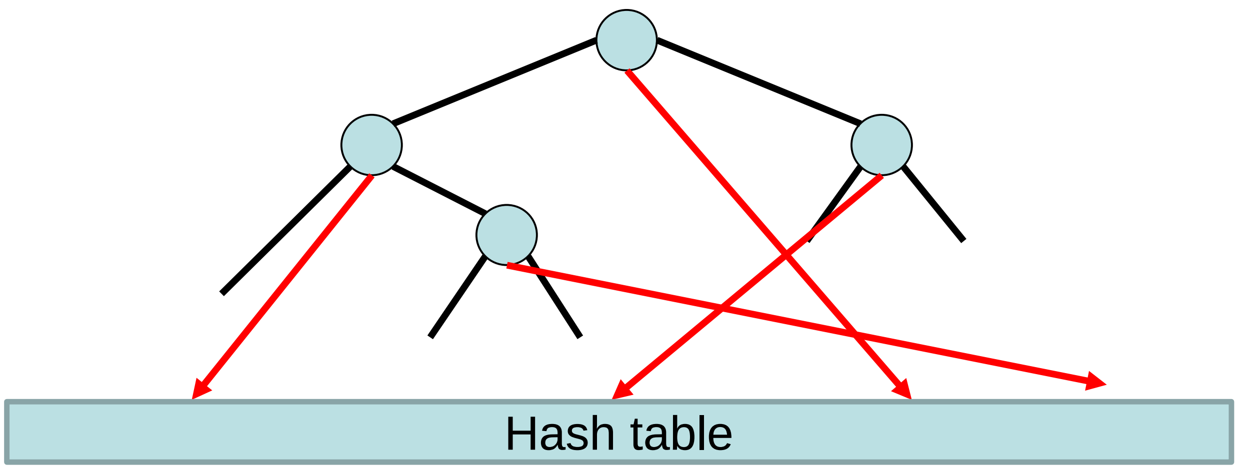


- Then every Patricia node can directly be accessed via a HT-lookup if its label is known.

Patricia Trie

Hashing to some hash table:

- Node label: concatenation of edge labels from root
- Every node is hashed according to its node label.

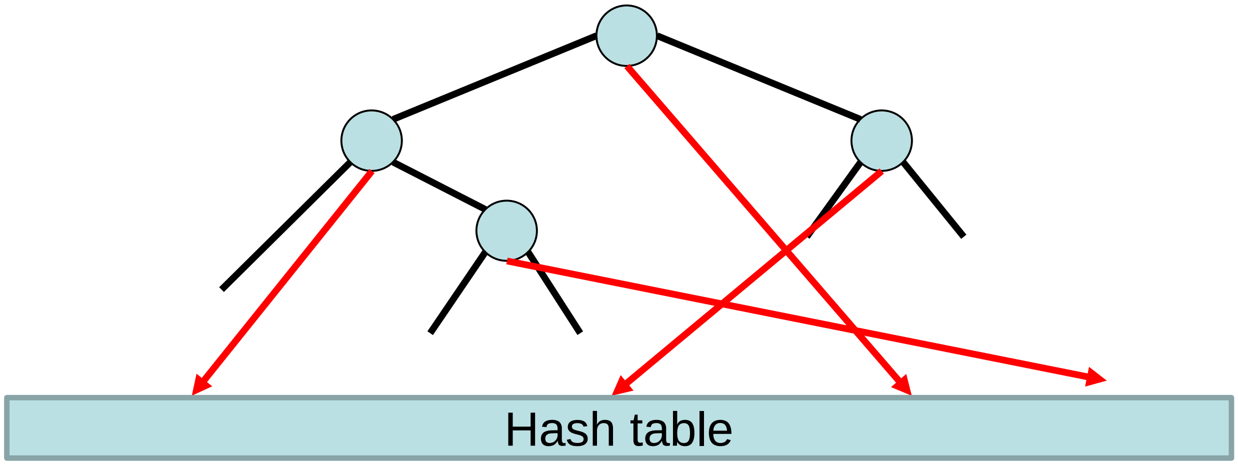


Problem: HT-lookups would, in principle, allow binary search on node labels (time O(W) → O(log W)) but not yet feasible.

Patricia Trie

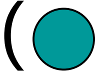
Hashing to some hash table:

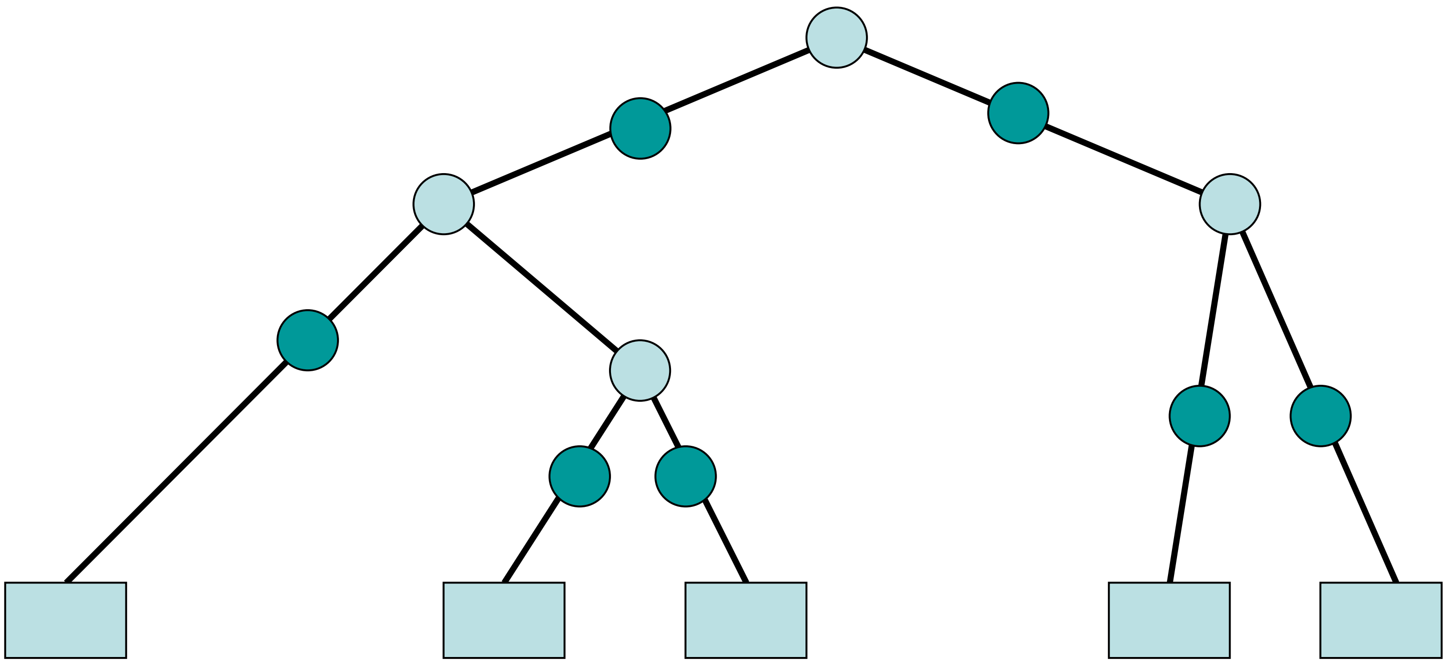
- Node label: concatenation of edge labels from root
- Every node is hashed according to its node label.



Solution: add extra support nodes (msd-nodes).

Patricia Trie Hashing

Solution: add msd-node () for each edge.



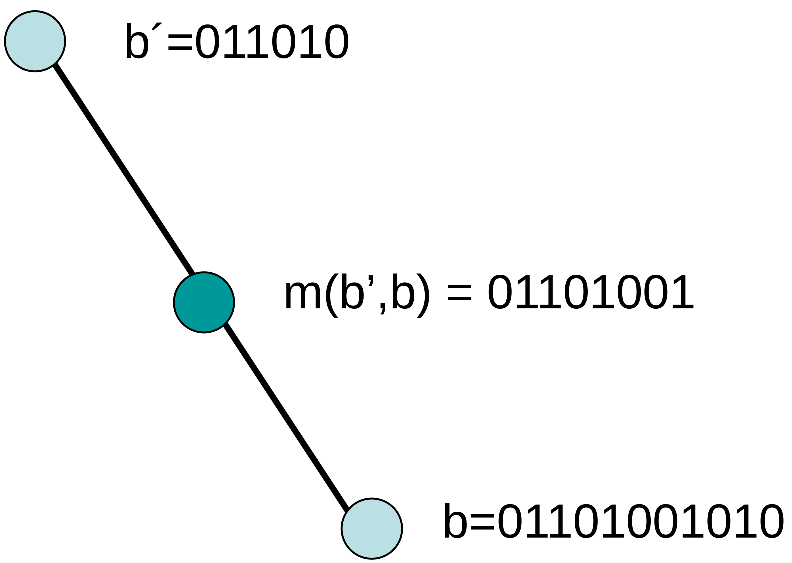
Patricia Trie Hashing

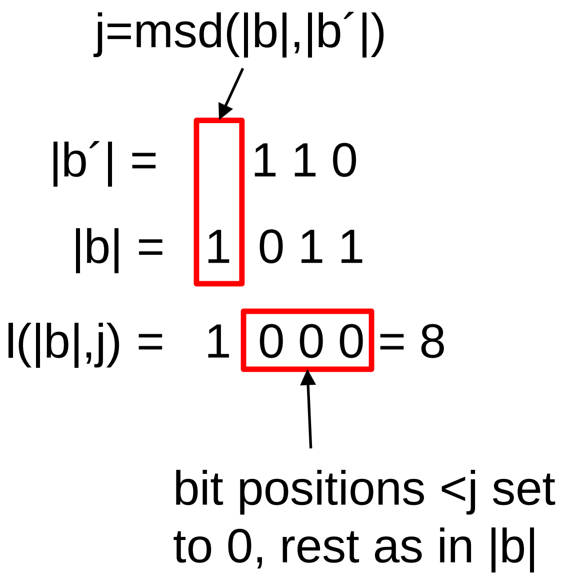
- $|x|$: length of a bit sequence x.
- $b(v)$: label of node v.
- msd(f,f´) for two bit sequences $0^* \circ f$ and $0^* \circ f´$: largest bit position (starting with position 0 from right) in which $0^* \circ f$ and $0^* \circ f´$ differ ($0^*=0000\ldots$).
- Consider a bit sequence b with $(x_k,\ldots,x_0)$ being the binary representation of $|b|$. Let b' be a prefix of b. The msd-sequence $m(b',b)$ of b' and b is the prefix of b of length $l(|b|,j)=\sum_{i=j}^{k} x_i\, 2^i$ with $j=msd(|b|,|b'|)$.

Example: Consider $b=01101001010$ and $b'=011010$. Then $|b|=11$, or in binary, 1011, and $|b'|=6$, or in binary, 110, i.e., $msd(|b|,|b'|)=3$. Hence, $m(b',b)= 01101001$.

Patricia Trie Hashing

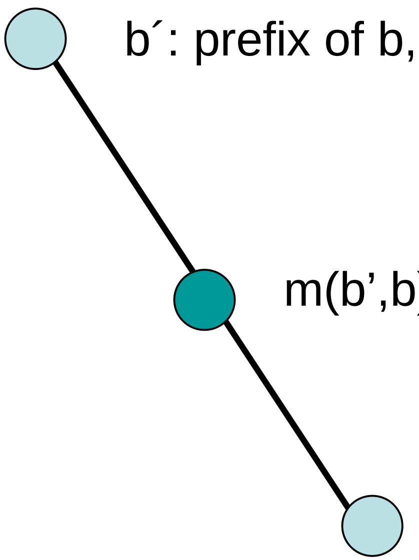
Example: Consider b=01101001010 and b'=011010. Then |b|=11, or in binary, 1011, and |b'|=6, or in binary, 110, i.e., msd(|b|,|b'|)=3. Hence, m(b',b)= 01101001.





Patricia Trie Hashing

Other example:



j=msd(|b|,|b´|)

|b´| = 1 0 **0** 1 0 0

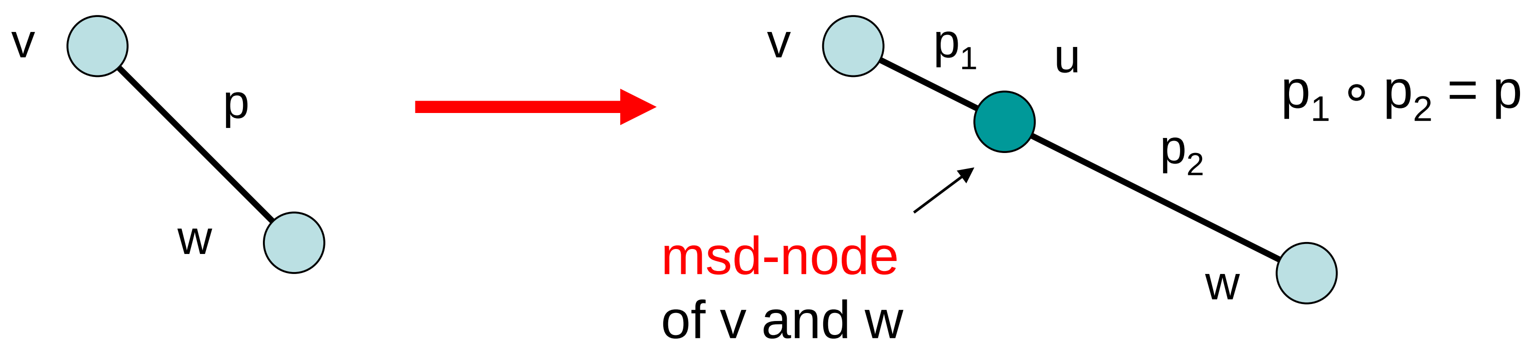
|b| = 1 0 **1** 0 1 1

l(|b|,j) = 1 0 1 **0 0 0** = 40

bit positions <j set to 0, rest as in |b|

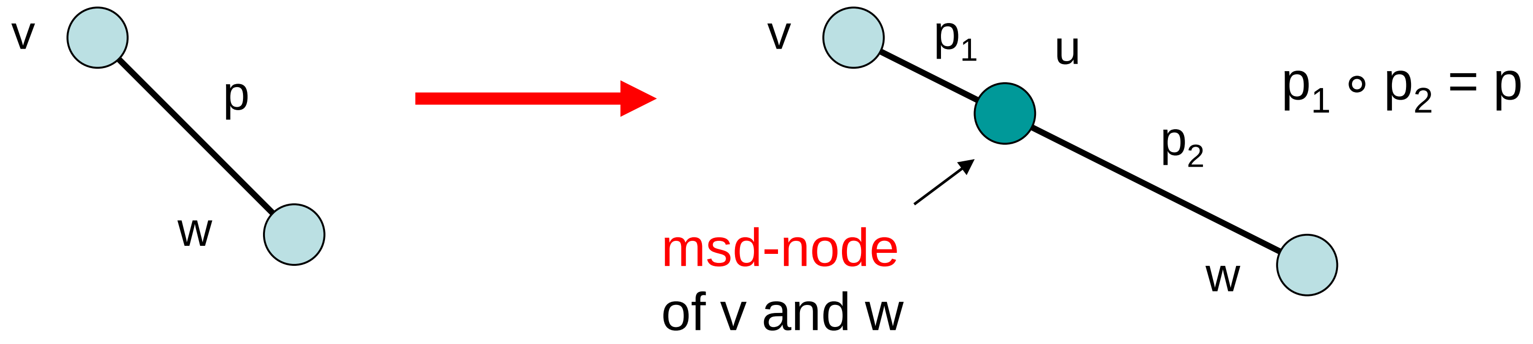
Patricia Trie Hashing

Approach: We replace every edge $e=\{v,w\}$ in the Patricia trie by two edges $\{v,u\}$ and $\{u,w\}$ with $b(u)=m(b(v),b(w))$ and hash the corresponding Patricia trie to the given hash table.

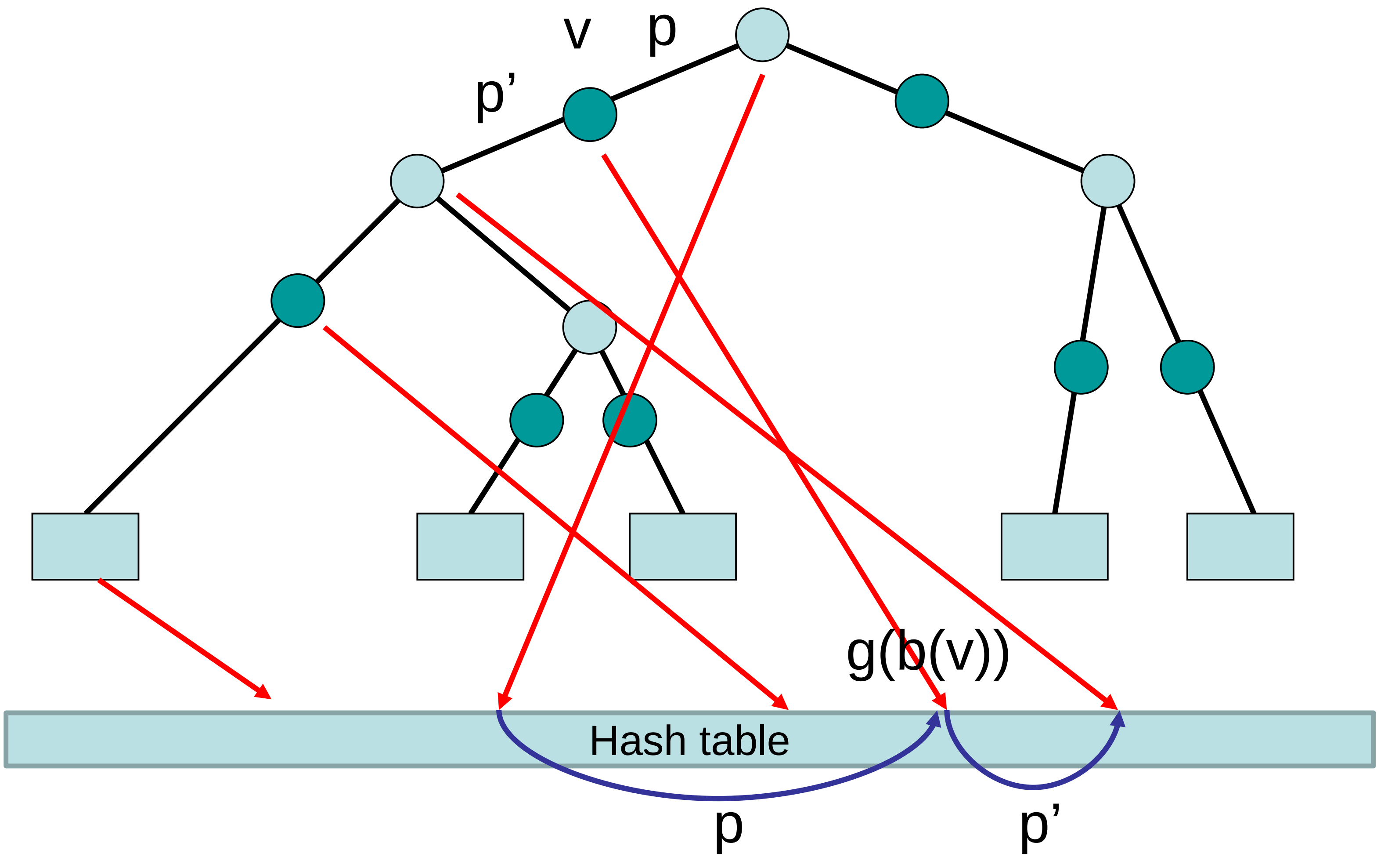


Patricia Trie Hashing

Motivation for inserting msd-nodes: msd-node placed at the position where binary search on the label length will look for the first time for a label of length between |b(v)| and |b(w)|.



Patricia Trie Hashing



Patricia Trie Hashing

Data structure for longest prefix search:

Every hash entry of a tree node v stores:

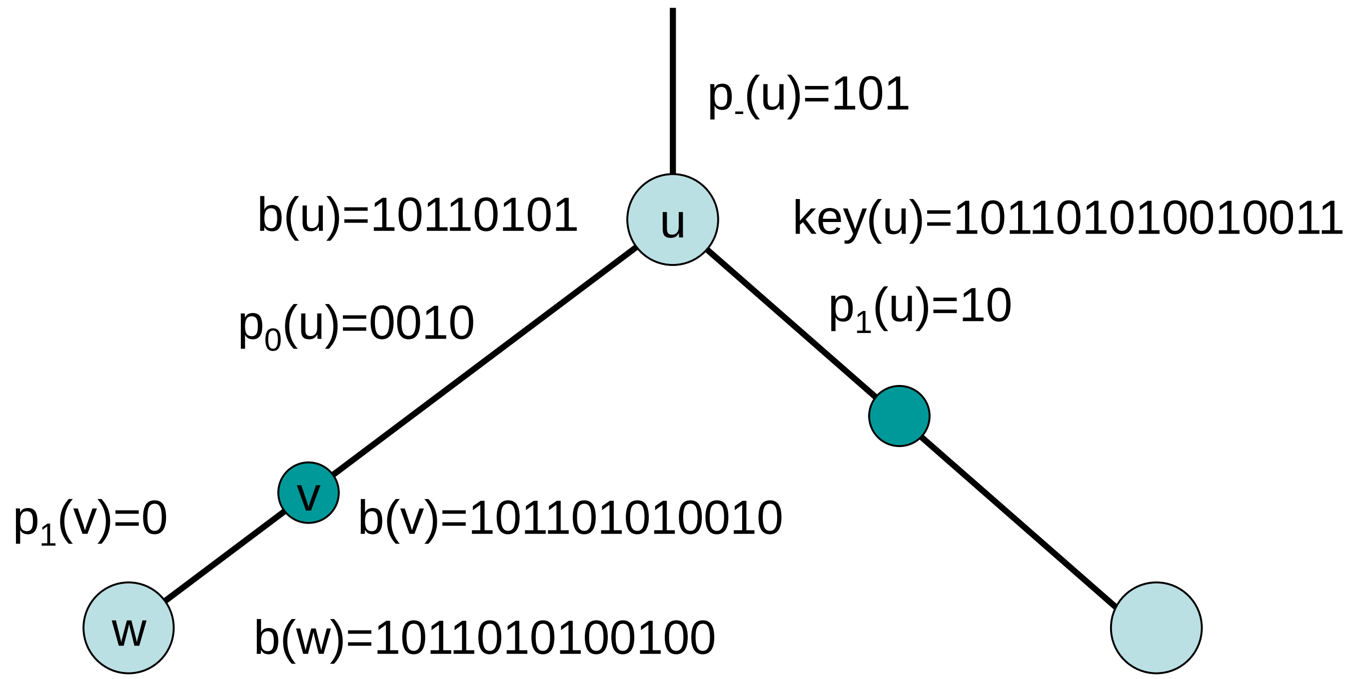
1. Label $b(v)$ of v (always ε for the root!)
2. Key $key(v)$ of an element e below the subtree of v, if v is an original Patricia trie node
3. Labels $p_x(v)$ of edges to children, $x \in \{0,1\}$
4. Label $p_-(v)$ of edge to parent (root: $p_-(v)=$prefix to root)

Every hash entry of a list element e stores:

1. Key of e
2. Label $p_-(v)$ of edge to parent
3. Label of tree node storing $key(e)$

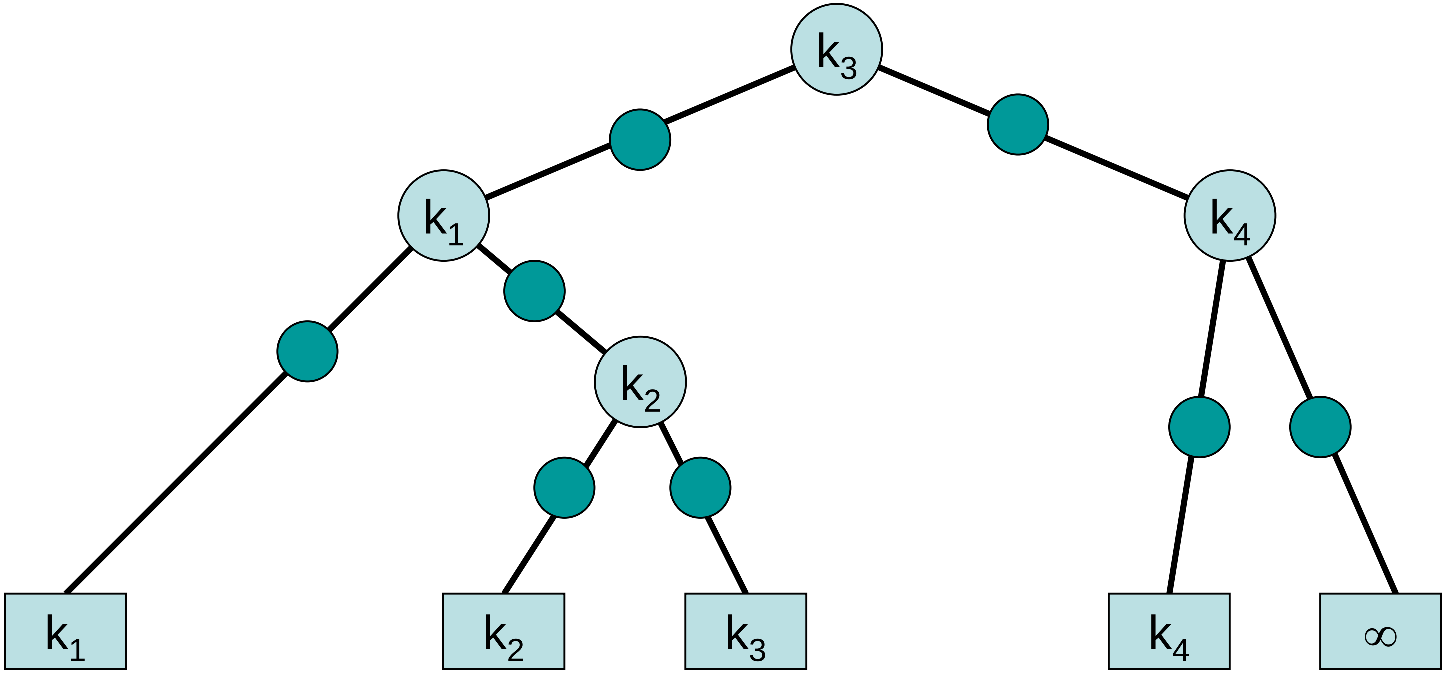
Patricia Trie Hashing

Example:



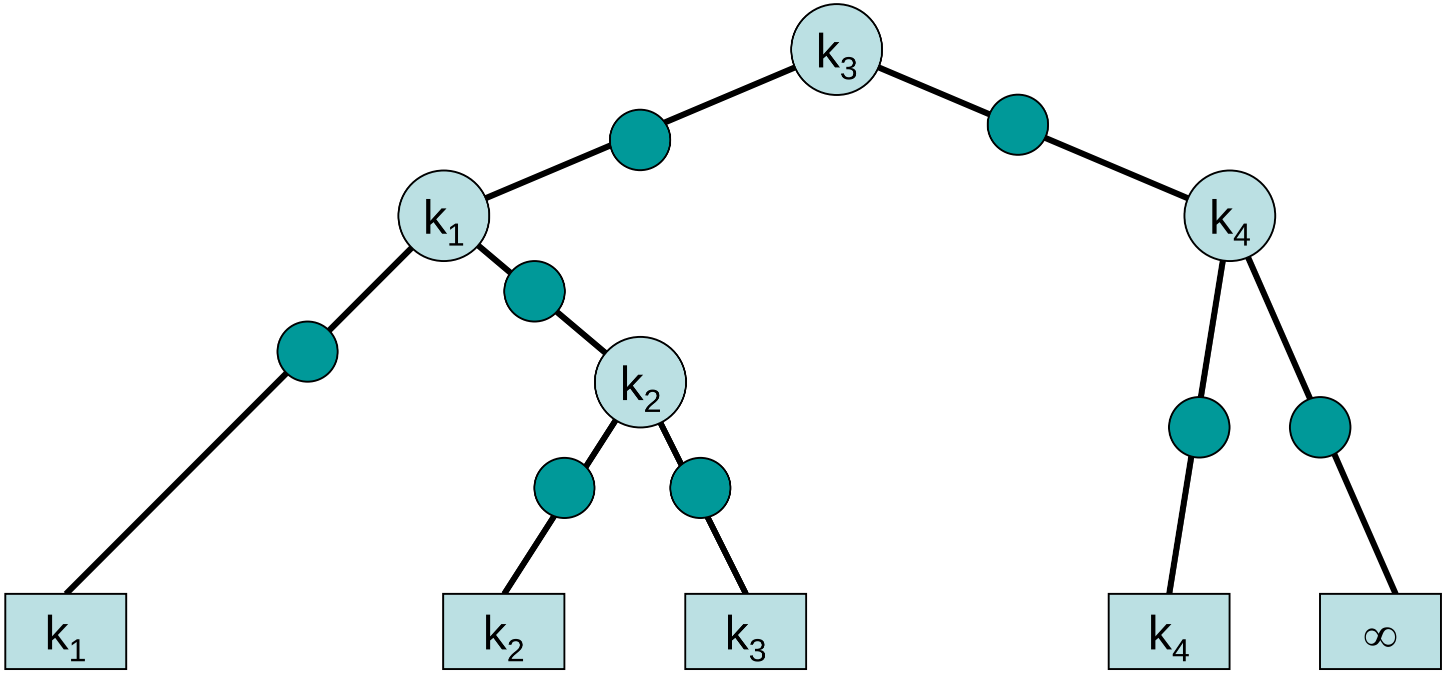
Patricia Trie Hashing

Requirement: every tree node stores key of exactly one element (possible with ∞).



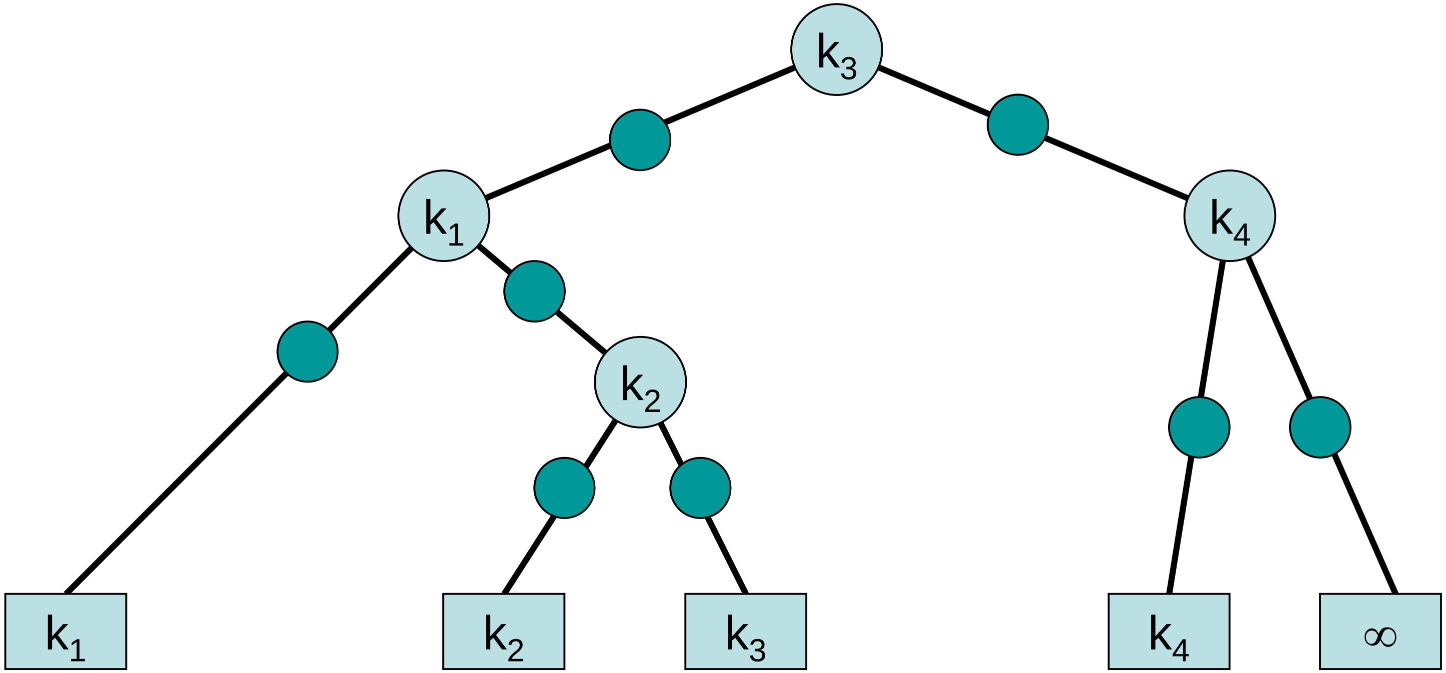
Patricia Trie Hashing

Invariant: the label of a tree node is a prefix of the key stored in it.



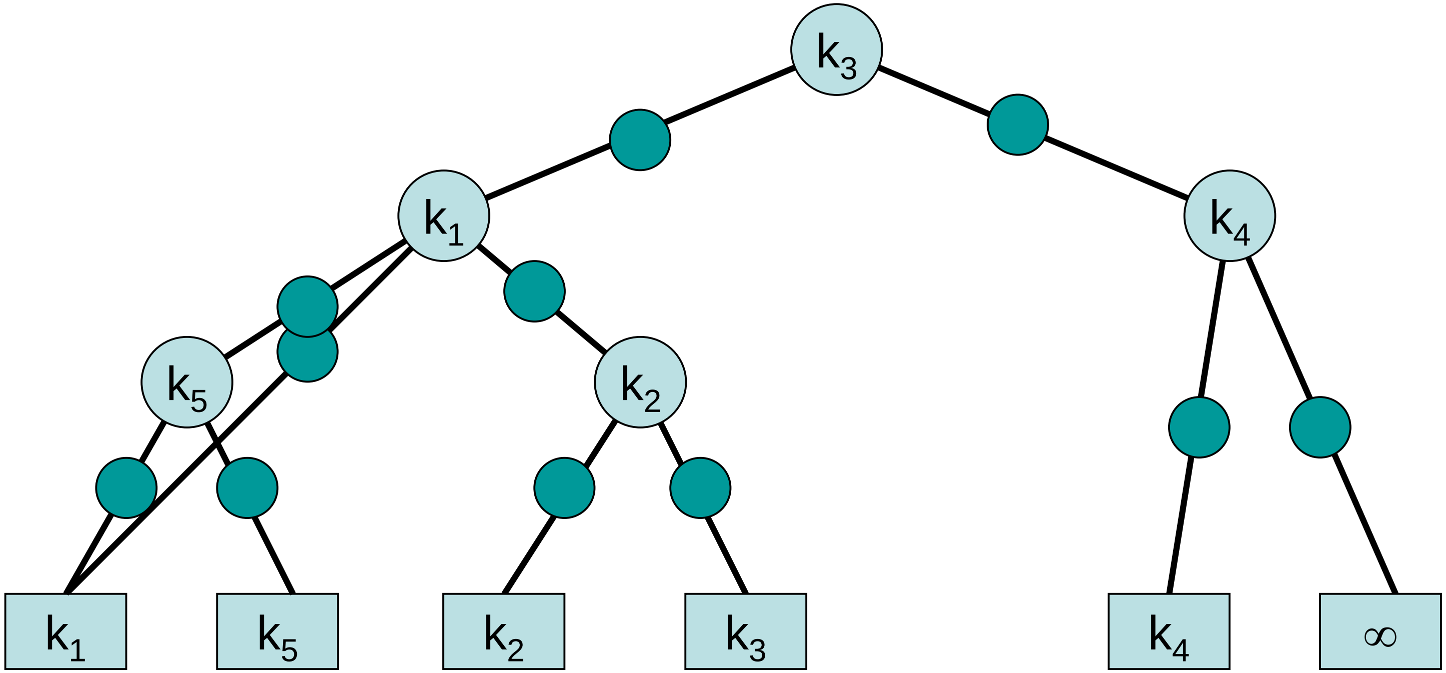
Patricia Trie Hashing

We first illustrate the structural changes for insert and delete.



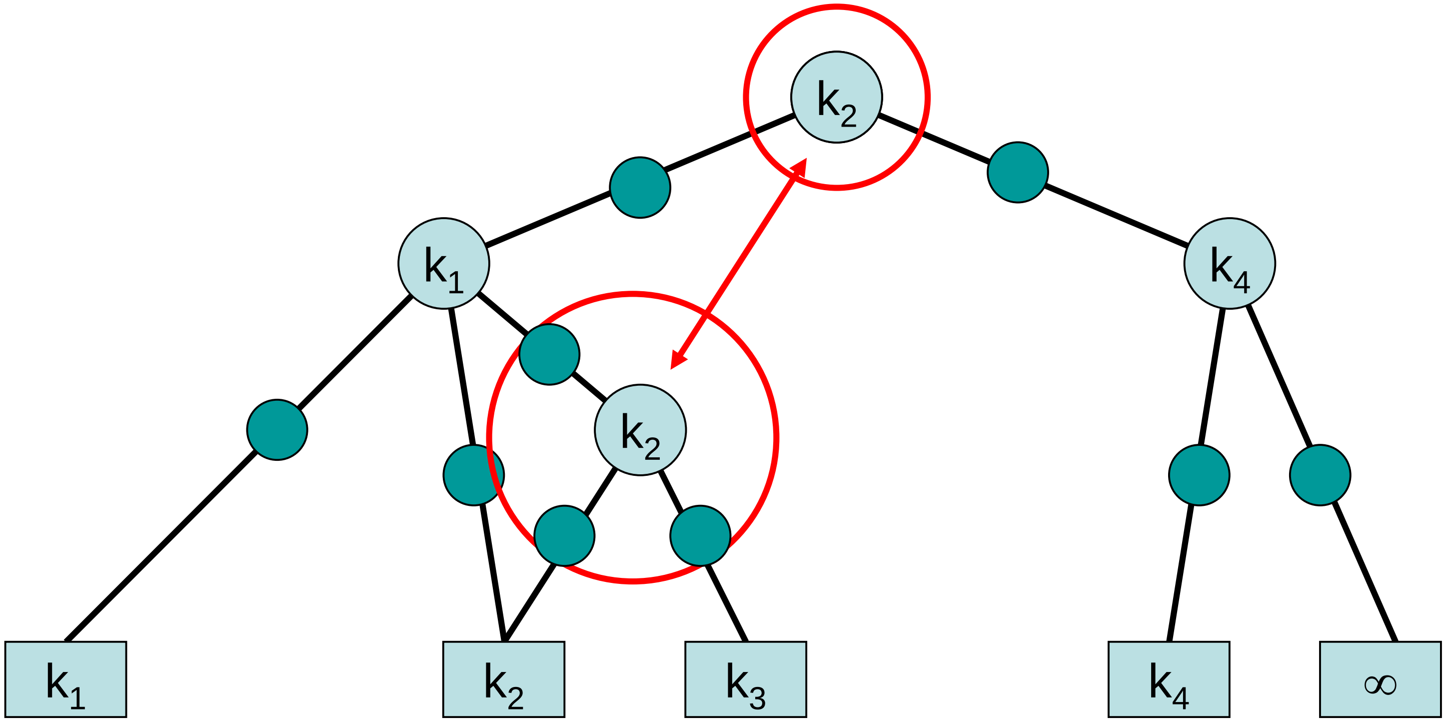
Patricia Trie Hashing

Insert(e), key(e)=k_5: like in binary search tree



Patricia Trie Hashing

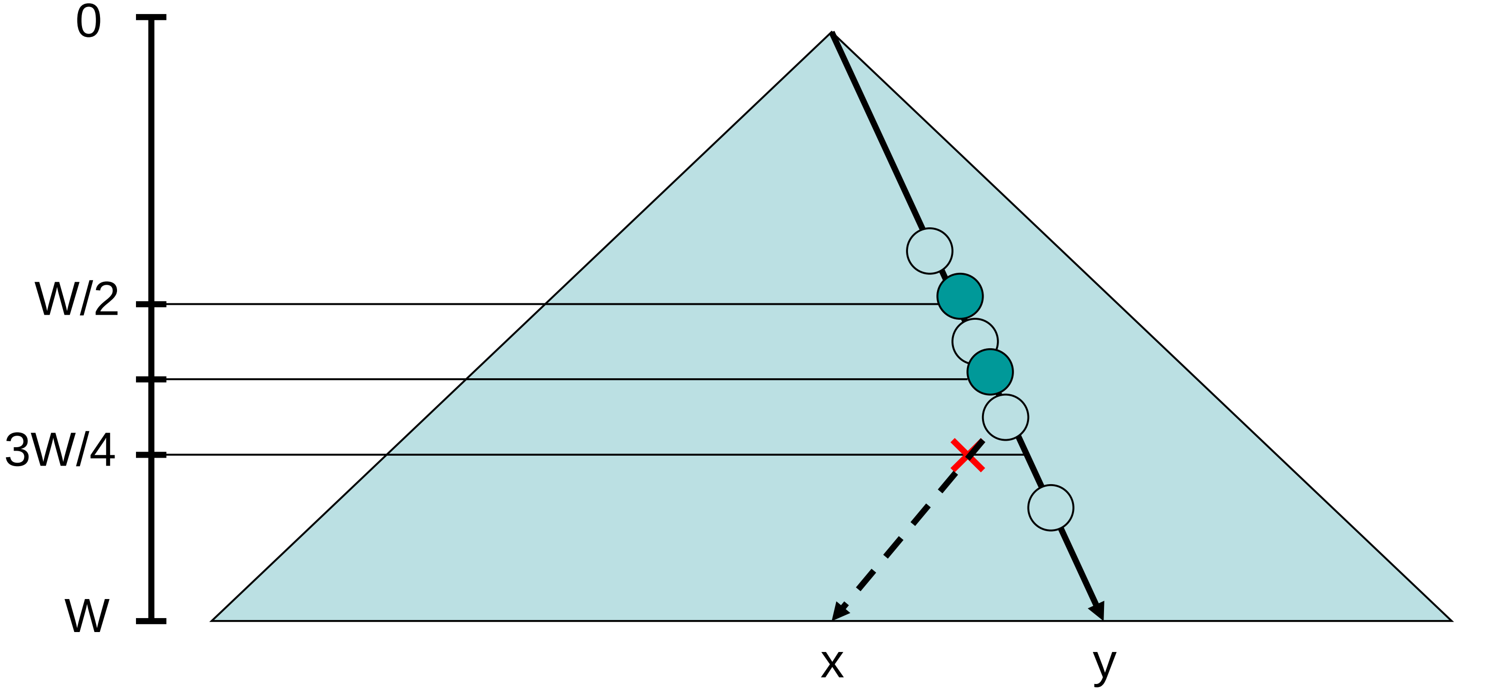
Delete(k_3): like in binary search tree



Patricia Trie Hashing

Search(x): (W: power of two)

Phase 1: binary search via msd-nodes

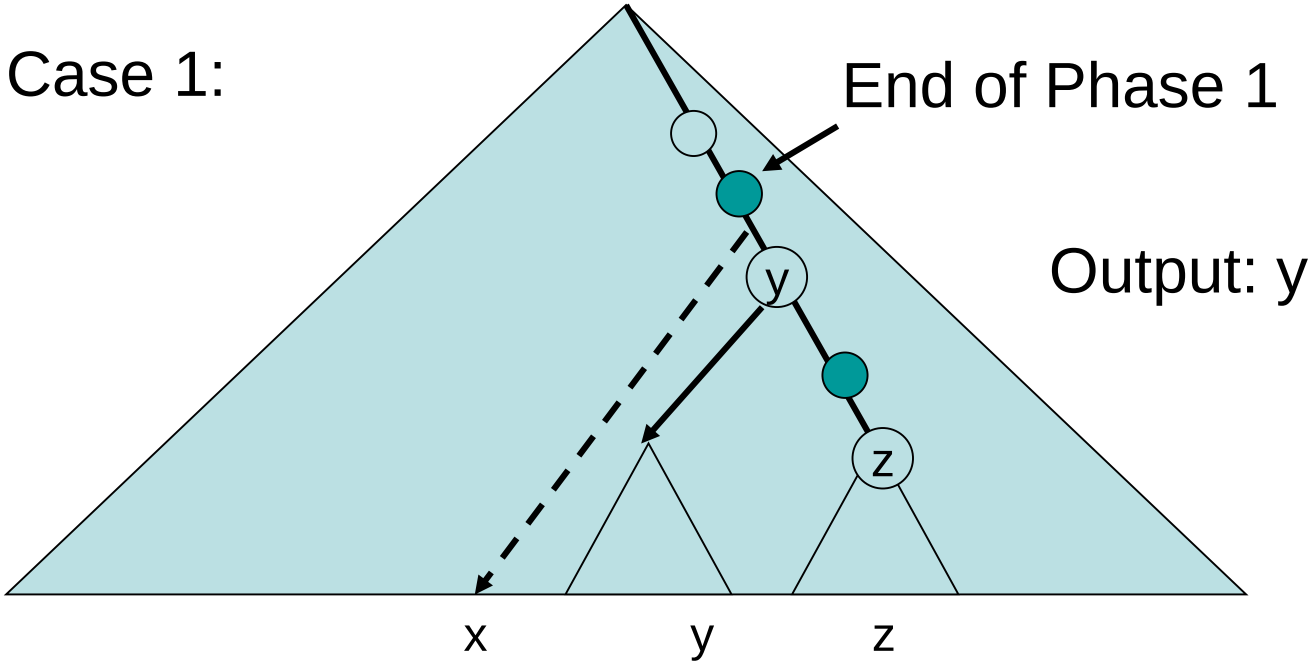


Patricia Trie Hashing

Search(x): (W: power of two)

Phase 2: read keys from tree node

Case 1:

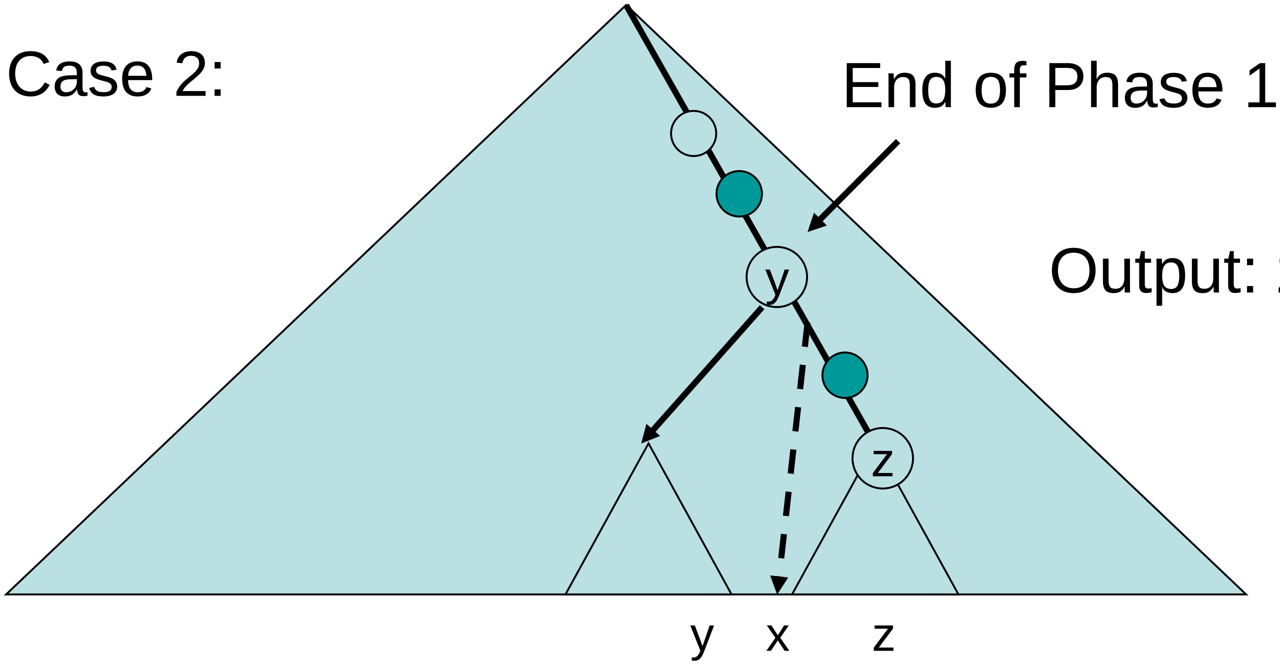


Patricia Trie Hashing

Search(x): (W: power of two)

Phase 2: read keys from tree node

Case 2:



Output: z

Patricia Trie Hashing

- Let $x \in \{0,1\}^W$ be represented by $(x_1,\ldots,x_W)$
- Hash function: $h:U \rightarrow [0,1)$

search(x):
    // found, then done
    if $key(T[h(x)])=x$ then return $T[h(x)]$
    // Phase 1: binary search for x
    $s:=\lfloor \log W \rfloor$; $k:=0$; $v:=T[h(\varepsilon)]$; $p:=p_{-}(v) \circ p_{x_1}(v)$ // v: root of Patricia trie
    while $s >= 0$ do
        // is there node with label $(x_1,\ldots,x_{k+2^s})$?
        if $(x_1,\ldots,x_{k+2^s}) = b(T[h(x_1,\ldots,x_{k+2^s})])$ // yes
            then $k:=k+2^s$; $v:=T[h(x_1,\ldots,x_k)]$; $p:= (x_1,\ldots,x_k) \circ p_{x_{k+1}}(v)$
            else if $(x_1,\ldots,x_{k+2^s})$ is prefix of p
                // edge from v covers $(x_1,\ldots,x_{k+2^s})$
                then $k:=k+2^s$
        $s:=s-1$
    // to be continued with Phase 2…

Patricia Trie Hashing

search(x): (continued from previous slide)

// Phase 1 stops at deepest v with b(v) being a prefix of $(x_1,\ldots,x_W)$

// Phase 2: find key of largest prefix

if $p_{x_{k+1}}(v)$ exists then

$v := T[h(b(v) \circ p_{x_{k+1}}(v))]$

else

$v := T[h(b(v) \circ p_{\bar{x}_{k+1}}(v))]$

if v is msd-node then

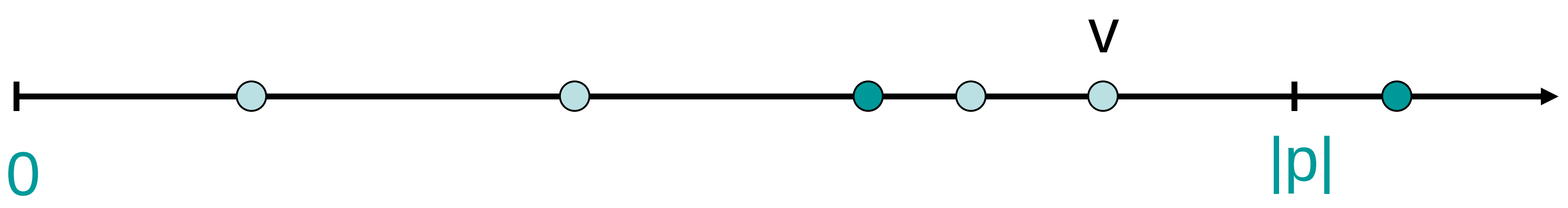
$v := T[h(b(v) \circ p)]$ for bit sequence p out of v

return key(v)

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\dots,z_0)$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node



Binary representation of $|b(v)|$ has ones at positions $i_1,i_2,\dots$ (i_1: maximal position)

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)$.
- Patricia trie contains a route for prefix p
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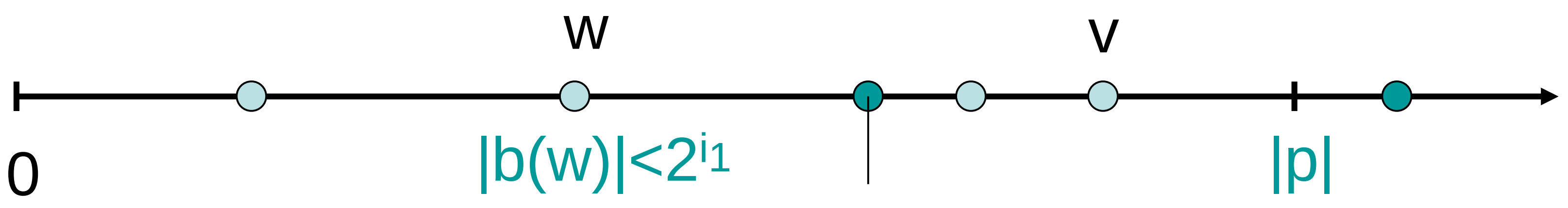
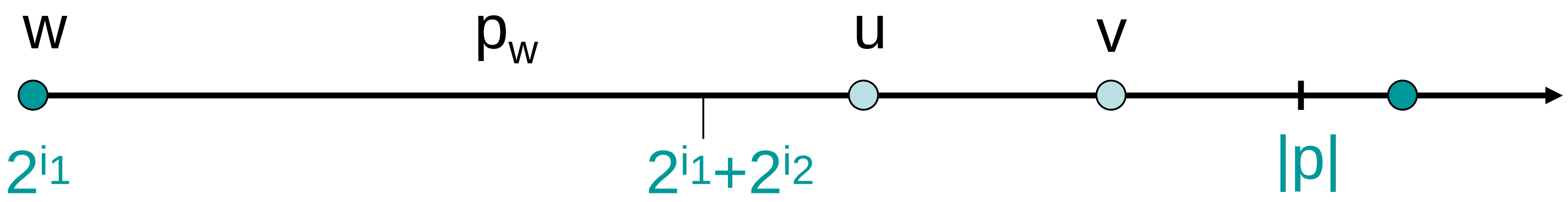


msd-node must exist at 2^{i_1},
will be found by binary search

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node

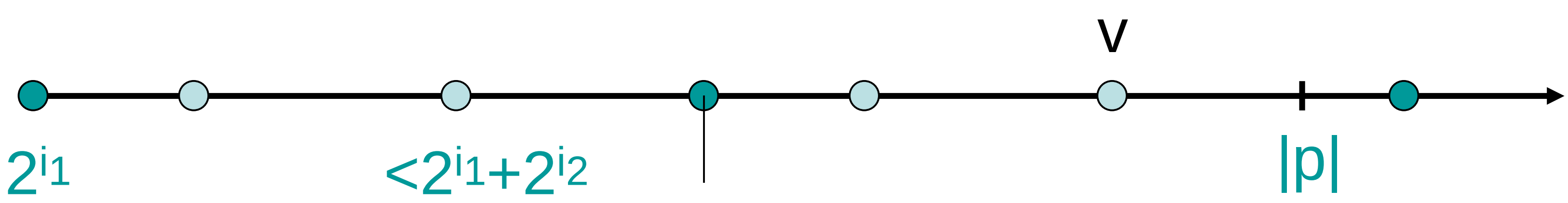


a) no msd-node at $2^{i_1}+2^{i_2}$: only if no Patricia node u with $2^{i_1}<|b(u)|\leq 2^{i_1}+2^{i_2}$, but this can be recognized via p_w

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\dots,z_0)$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node

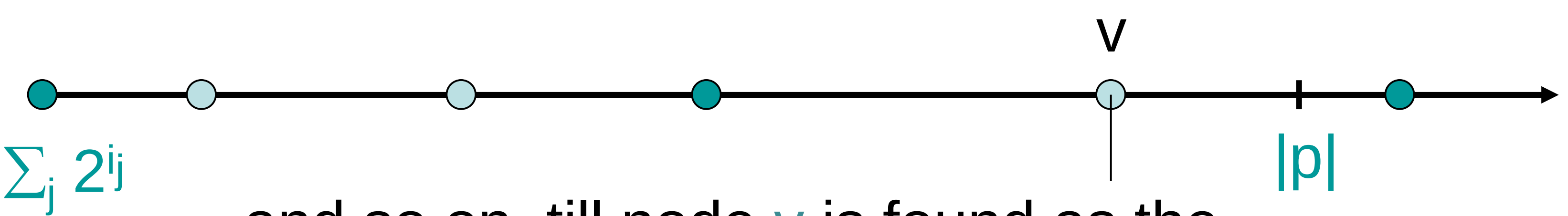


b) msd-node at $2^{i_1}+2^{i_2}$: is found by binary search

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\ldots,z_0)$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 1: v is Patricia node

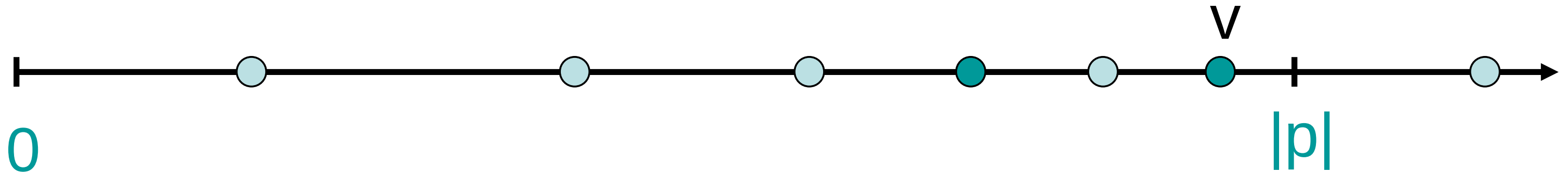


and so on, till node v is found as the last node of the binary search

Patricia Trie Hashing

Correctness of phase 1:

- Let p be largest common prefix of x and an element $y \in S$ and let $|p|=(z_k,\dots,z_0)$.
- Patricia trie contains a route for prefix p
- Let v be last node on route till p
- Case 2: v is msd-node



v will also be the last node of binary search
if it is an msd-node (argue like in case 1)

Patricia Trie Hashing

Number of HT accesses for longest prefix search:

- O(log W) HT-lookups, where W is key length

Number of HT accesses for insert:

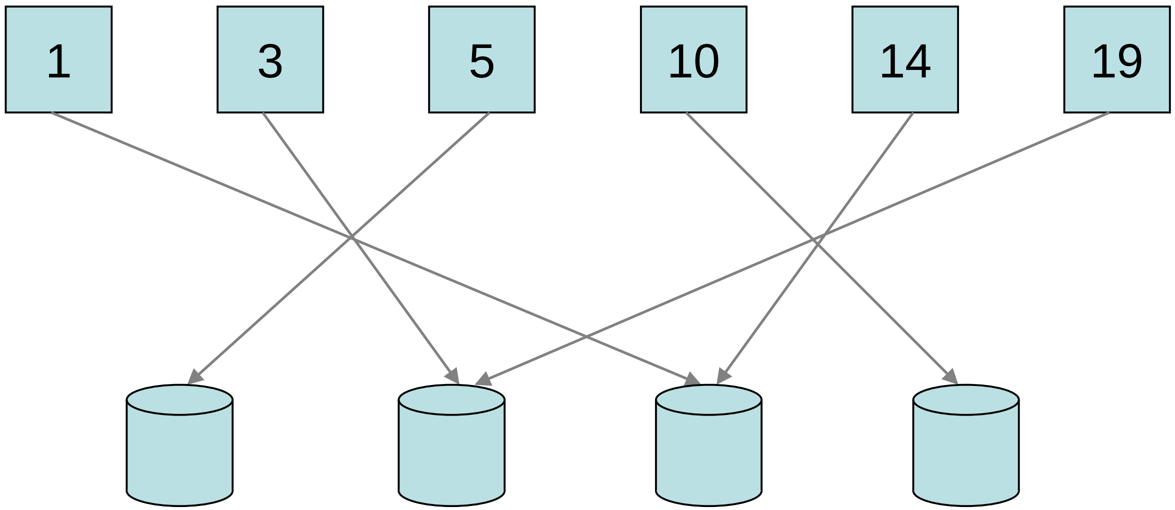
- O(log W) HT-lookups
- O(1) HT-updates

Number of HT accesses for delete:

- O(1) HT-lookups
- O(1) HT-updates

Patricia Trie Hashing

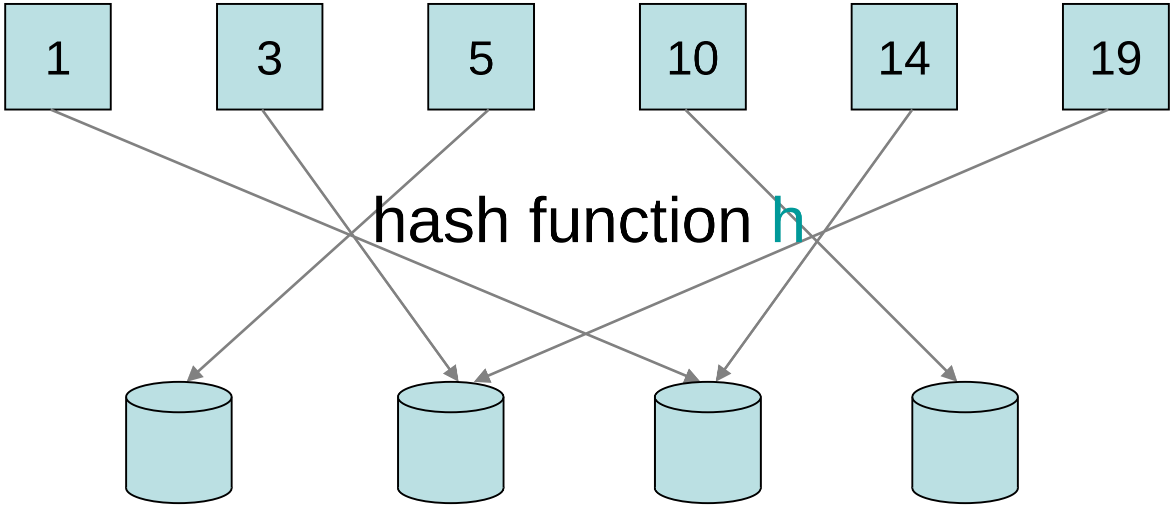
Application: distributed storage system



Goal: minimize number of accesses to servers for longest prefix match

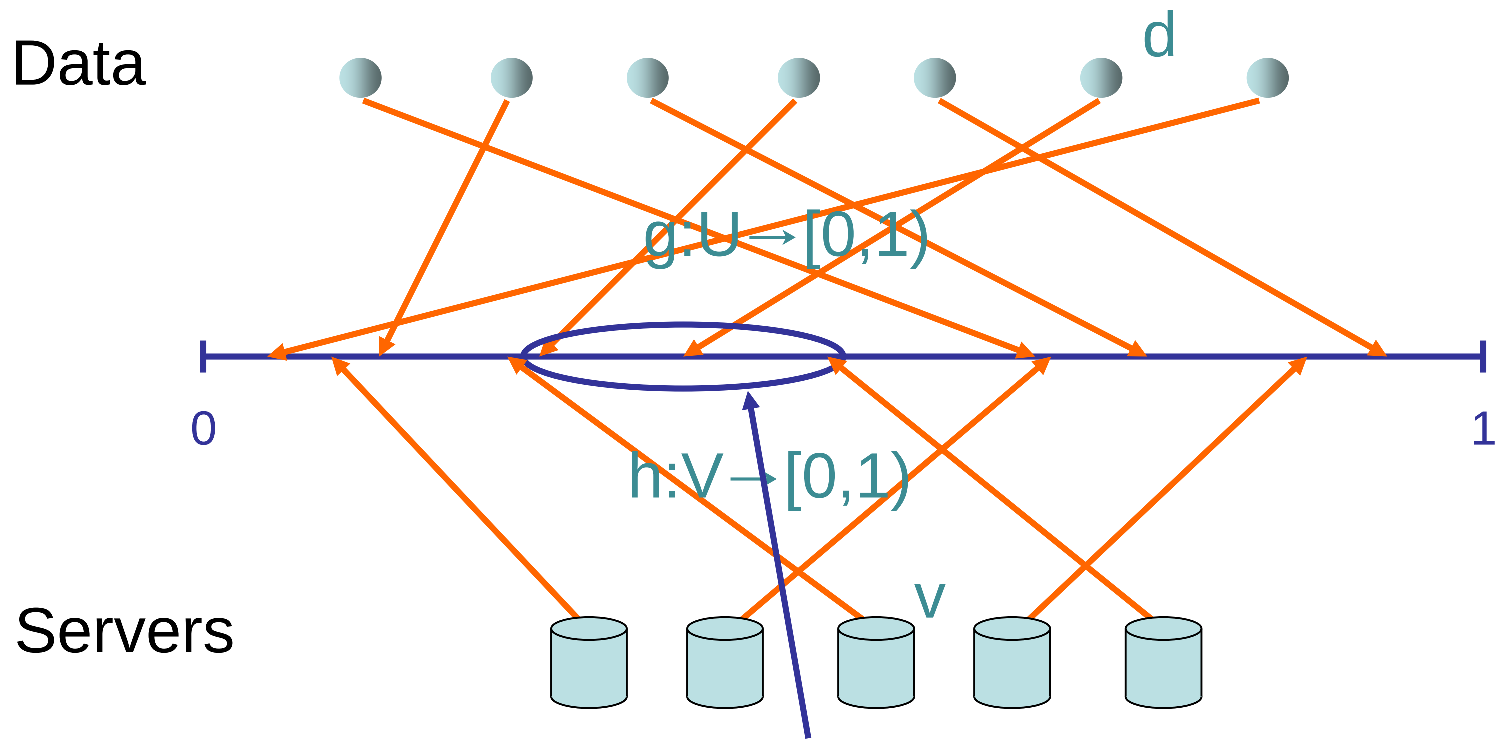
Distributed Storage System

Standard approach for exact search:
distributed hash table (DHT)



Consistent Hashing

Choose two random hash functions h, g



Region that server v is responsible for

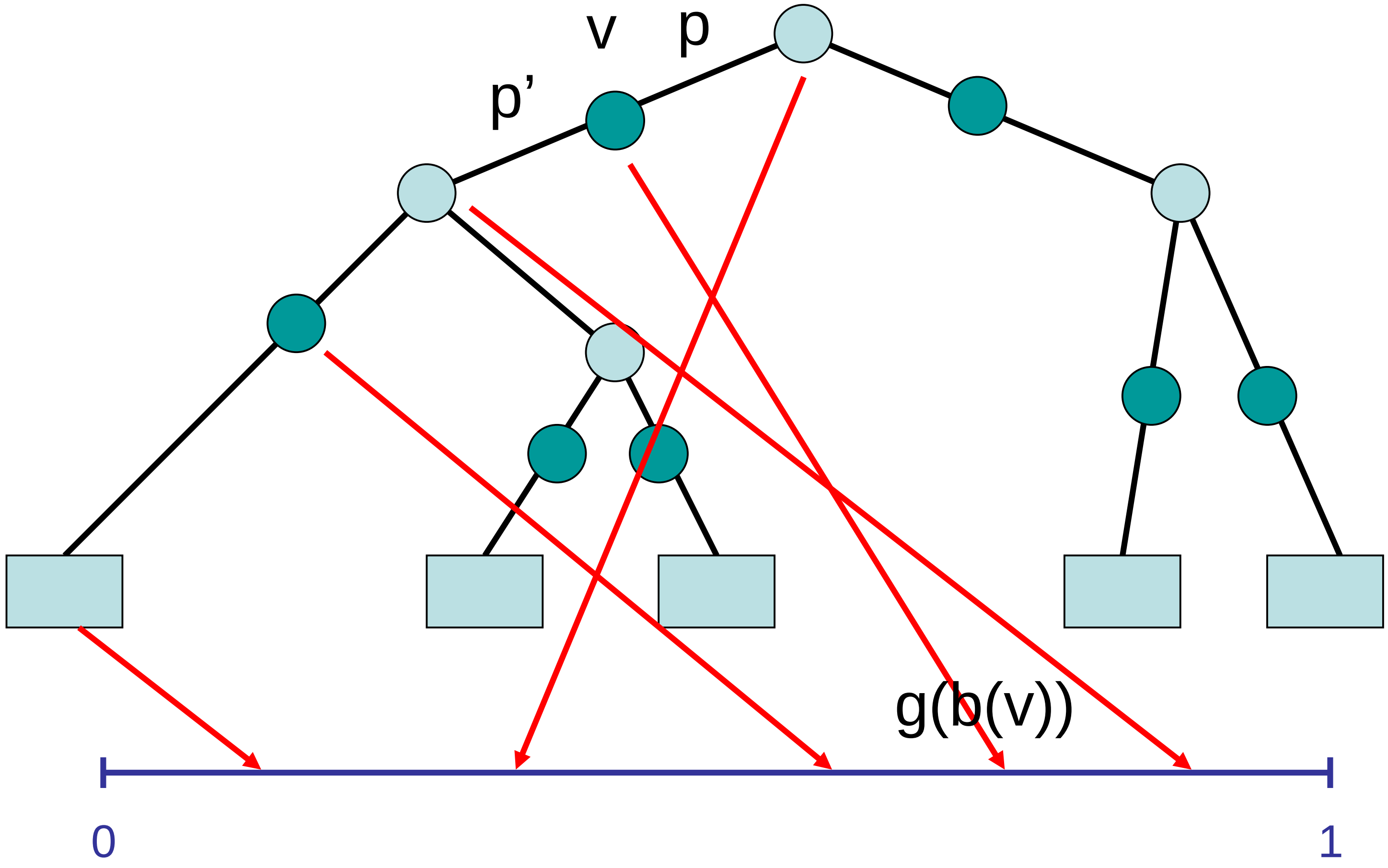
Consistent Hashing

- V: current set of servers
- succ(v): closest successor of v in V w.r.t. hash function h (where [0,1) is viewed as a cycle)
- pred(v): closest predecessor of v in V w.r.t. h

Assignment rules:

- One copy per data item: server v stores all items d with $g(d) \in I(v)$, where $I(v)=[h(v), h(succ(v)))$.
- $k>1$ copies per data item: d is stored in the above server v and its $k-1$ closest successors w.r.t. h

Distributed Patricia Trie Hashing



Distributed Patricia Trie Hashing

Number of DHT accesses for longest prefix search:

- O(log W), where W is key length

Number of DHT accesses for insert:

- O(log W) for lookups
- O(1) for updates

Number of DHT accesses for delete:

- O(1) for lookups
- O(1) for updates